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# TENSION OF A CYLINDRICAL BAR HAVING AN INFINITE ROW OF CIRCUMFERENTIAL CRACKS

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**Abstract**—In this paper, the crack problems in the case of a cylindrical bar having a circumferential crack and a cylindrical bar having an infinite row of circumferential cracks under tension are analyzed by the body force method. The stress field for a periodic array of ring forces in an infinite body is used to solve the problems. The solution is obtained by superposing the stress fields of ring forces in order to satisfy a given boundary condition. The stress intensity factors are calculated for various geometrical conditions. The obtained values of stress intensity factor of a single circumferential crack are considered to be more reliable than the results of other paper's. As the crack becomes very shallow, the stress intensity factor of a row of circumferential cracks approaches the value corresponding to that of a row of edge cracks in a semi-infinite plate under tension. As the crack becomes very deep, it approaches the values corresponding to that of a single deep circumferential crack.

## NOTATION

$c$ crack depth
$D$ cylindrical diameter
$d$ diameter of a minimum section
$h$ pitch of cracks
$\nu$ Poisson's ratio
$(r, \theta, z)$ cylindrical coordinates of a point in question
$(\rho, \phi, \xi)$ cylindrical coordinates of a point where a point force acts
$P_r, P_z$ strength of a point force
$P_{DB}$ strength of a pair of point forces
$F_r, F_z$ strength of a ring force (force per unit length)
$F_{DB}$ strength of a pair of ring forces
$\rho_r, \rho_z$ density of body force (force per unit area)
$\rho_{DB}$ density of a pair of body forces
$\sigma^{P_r}, \sigma^{P_z}$ stresses due to a point forces
$\sigma^{P_{DB}}$ stresses due to a pair of point forces
$\sigma^{F_r*}, \sigma^{F_z*}$ stresses due to a ring force
$\sigma^{F_{DB}*}$ stresses due to a pair of ring forces
$\sigma^{F_r**}, \sigma^{F_z**}$ stresses due to a periodic array of ring forces
$\sigma^{F_{DB}*}$ stresses due to a periodic array of pairs of ring forces
$\sigma_z^\infty$ nominal stress for the cylindrical diameter $D$
$\sigma_{net}$ nominal stress for the minimum diameter $d$
$i$ number of the interval in question
$j$ number of the interval where body force is applied
$\sigma^{\rho_{rj}}, \sigma^{\rho_{zj}}; \sigma^{\rho_{DB}}$ influence coefficients, which mean the stresses induced at the midpoint of $i$th interval by the unit body force acting at the $j$th interval
$n_1$ division number of a crack surface
$n_2$ division number of a cylindrical surface
$n_t$ total division number $(= n_1 + n_2)$
$K_I$ stress intensity factor
$F_I$ dimensionless stress intensity factor; $K_I = F_I \sigma_z^\infty \sqrt{\pi c}$
$F_I'$ dimensionless stress intensity factor; $K_I = F_I' \frac{1}{2}\sigma_{net}\sqrt{\pi d/2}$

## 1. INTRODUCTION

A CYLINDRICAL bar having a circumferential crack shown in Fig. 1 has been used as a specimen which determines the fracture toughness of materials. Therefore, many researchers have tried to obtain the stress intensity factor of this problem [1–4]. Benthem–Koiter [2] have proposed an approximate formula which gives the exact values in the limiting two cases, a very deep crack and a very shallow crack. In 1977, Keer–Freedmann–Watts [3] have obtained the accurate stress intensity factors for a wide range of crack depth by using integral transform technique and solving singular integral equations numerically. Atsumi–Shindo [4] extended the method of analysis by Keer *et al.* and studied the problem of a transversely isotropic cylindrical bar having a circumferential crack. An approximate formula by Benthem–Koiter is convenient for practical use, but the error of this formula should be estimated by comparing its value with the exact solution.

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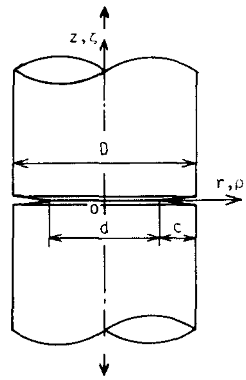


Fig. 1. Tension of a cylindrical bar having a circumferential crack.

The problem of a cylindrical bar having an infinite row of circumferential cracks shown in Fig. 2 is useful in investigating the interference effect among cracks. Although there are many reports concerning the problem of a single circumferential crack, the research of the interference effect among circumferential cracks in the tension of a cylindrical bar is not found.

In this paper, the crack problem in the case of a cylindrical bar having a circumferential crack (Fig. 1) or an infinite row of circumferential cracks (Fig. 2) under tension are analyzed by the body force method [5, 6].

## 2. METHOD OF ANALYSIS

The body force method for solving three-dimensional axisymmetric problems is based on using the stress field of ring forces in an infinite body. The ring forces acting in the $r$-direction ($F_r$) or in the $z$-direction ($F_z$) shown in Fig. 3 are used for the analysis of tension problems.

In the present analysis, the boundary condition of a cylindrical surface are satisfied by applying body forces (continuously embedded ring forces) along the prospective boundary imagined in an infinite body. On the other hand, the boundary conditions of a circumferential crack are satisfied by applying a pair of body forces (continuously embedded pairs of ring forces) [5]. The method of analysis in this problem is reduced to determining the densities of body force and a pair of body forces. The fundamental stress fields in this case are the stress field due to a ring force and the derivative of it.

As the definition of the density of a pair of body forces $\rho_{DB}$, the following expression is used [5].

$$\rho_{DB} = \frac{1-2\nu}{(1-\nu)^2} \frac{1}{4\sqrt{c^2-(\rho-D/2)^2}} \frac{dG_\zeta}{\rho d\rho d\phi} \tag{1}$$

in which $dG_\zeta$ is "the strength of a pair of body forces" acting at the element $\rho d\rho d\phi$, and ($\rho$, $\phi$, $\zeta$) are the cylindrical coordinates of a point where the pair of body forces acts. "A pair of point

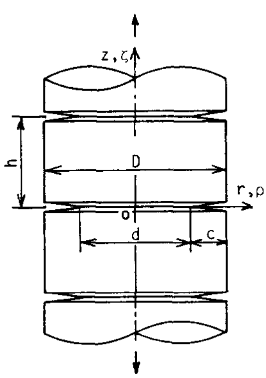


Fig. 2. Tension of a cylindrical bar having an infinite row of circumferential cracks.

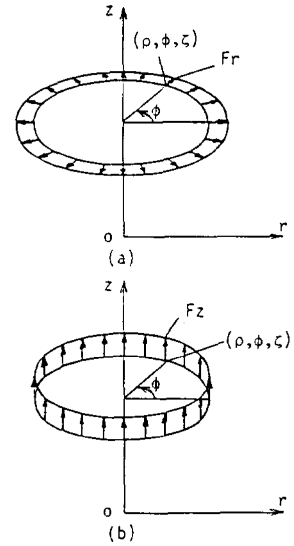


Fig. 3. A ring force acting $r$- or $z$-direction in an infinite body.

forces having unit strength" means the combination of three kinds of pairs of point forces, each acting in $r$-, $\theta$- and $z$-directions shown in Fig. 4 [7, 8]. In the axisymmetric problem, the ring of pairs of point forces acting in the $\theta$-direction does not affect the stress field because the effects of the pairs are cancelled with each other by integration. According to the definition of eqn (1), the value of $\rho_{DB}$ for the problem of a two-dimensional through crack in an infinite body is constant along the prospective site of the crack which must be free from stresses. If the definition of eqn (1) is used for a general problem, the variation of $\rho_{DB}$ along the prospective site of crack is small and the obtained solution is very accurate [6].

## 3. FUNDAMENTAL SOLUTIONS

The interference effect of an infinite row of circumferential cracks is studied in this paper. Therefore, it is necessary to obtain the two kinds of fundamental solutions; one is the stress field for a periodic array of pairs of ring forces $\sigma^{F_{DB}**}$ (for cracks), and the other is the stress field for a periodic array of ring forces $\sigma^{F_r**}$, $\sigma^{F_z**}$ (for cylindrical surfaces).

When a pair of point forces or a point force acts at a point $(\rho, \phi, \zeta + mh)$ in an infinite body, the stresses at $(r, 0, z)$ are given by eqn (2). In eqn (2), $P_{DB}$ means the strength of a pair of point forces and $P_r$, $P_z$ mean the strength of a point force.

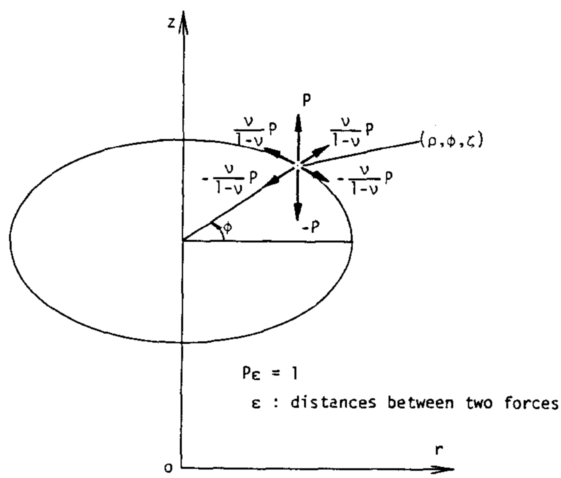


Fig. 4. A pair of point forces having unit strength.

$$\left.\begin{aligned}
\sigma_r^{P_{DB}} &= B_{DB}[2(1-\nu)R^{-3} + 3(1-2\nu)R^{-5}\rho^2(-1+\cos^2\phi) \\
&\quad + 15\bar{z}^2R^{-7}(-r^2 + 2r\rho\cos\phi - \rho^2\cos^2\phi)] \\
\sigma_\theta^{P_{DB}} &= B_{DB}[2(1-\nu)R^{-3} + 3(1-2\nu)R^{-5}(-r^2 + 2r\rho\cos\phi - \rho^2\cos^2\phi) \\
&\quad + 15\bar{z}^2R^{-7}\rho^2(-1+\cos^2\phi)] \\
\sigma_z^{P_{DB}} &= B_{DB}[R^{-3} + 6\bar{z}^2R^{-5} + 15\bar{z}^3R^{-7}(-r+\rho\cos\phi)] \\
\tau_{rz}^{P_{DB}} &= B_{DB}[3\bar{z}R^{-5}(r-\rho\cos\phi) + 15\bar{z}^3R^{-7}(-r+\rho\cos\phi)] \\
\sigma_r^{P_r} &= B_r[(1-2\nu)R^{-3}(-\rho - r\cos\phi + 2\rho\cos^2\phi) \\
&\quad + 3R^{-5}\{r^2\rho - r(r^2+2\rho^2)\cos\phi + \rho(2r^2+\rho^2)\cos^2\phi - r\rho^2\cos^3\phi\}] \\
\sigma_\theta^{P_r} &= B_r[(1-2\nu)R^{-3}(\rho + r\cos\phi - 2\rho\cos^2\phi) \\
&\quad + 3R^{-5}(\rho^3 - r\rho^2\cos\phi - \rho^3\cos^2\phi + r\rho^2\cos^3\phi)] \\
\sigma_z^{P_r} &= B_r[(1-2\nu)R^{-3}(-\rho + r\cos\phi) + 3\bar{z}^2R^{-5}(\rho - r\cos\phi)] \\
\tau_{rz}^{P_r} &= B_r[(1-2\nu)\bar{z}R^{-3}(-\cos\phi) + 3\bar{z}R^{-5}\{r\rho - (r^2+\rho^2)\cos\phi + r\rho\cos^2\phi\}] \\
\sigma_r^{P_z} &= B_z[(1-2\nu)\bar{z}R^{-3} + 3\bar{z}R^{-5}(-r^2 + 2r\rho\cos\phi - \rho^2\cos^2\phi)] \\
\sigma_\theta^{P_z} &= B_z[(1-2\nu)\bar{z}R^{-3} + 3\bar{z}R^{-5}(-\rho^2 + \rho^2\cos\phi)] \\
\sigma_z^{P_z} &= B_z[(1-2\nu)\bar{z}R^{-3}(-1) + 3\bar{z}^3R^{-5}(-1)] \\
\tau_{rz}^{P_z} &= B_z[(1-2\nu)R^{-3}(-r+\rho\cos\phi) + 3\bar{z}^2R^{-5}(-r+\rho\cos\phi)]
\end{aligned}\right\} \quad (2)$$

where

$$B_{DB} = \frac{P_{DB}}{8\pi(1-\nu)}, \quad B_r = \frac{P_r}{8\pi(1-\nu)}, \quad B_z = \frac{P_z}{8\pi(1-\nu)}$$

$$\bar{z} = z - \zeta - mh, \quad R^2 = r^2 + \rho^2 - 2r\rho\cos\phi + (z - \zeta - mh)^2.$$

Using eqn (2), the fundamental stress fields $\sigma^{F_{DB}**}$, $\sigma^{F_r**}$, $\sigma^{F_z**}$ can be expressed as follows.

$$\left.\begin{aligned}
\sigma^{F_{DB}**} &= \int_0^{2\pi}\sum_{m=-\infty}^{\infty}\sigma^{P_{DB}}\Big|_{P_{DB}=1} \times F_{DB}\rho\,d\phi = \sum_{m=-\infty}^{\infty}\int_0^{2\pi}\sigma^{P_{DB}}\Big|_{P_{DB}=1} \times F_{DB}\rho\,d\phi = \sum_{m=-\infty}^{\infty}\sigma^{F_{DB}*} \\
\sigma^{F_r**} &= \int_0^{2\pi}\sum_{m=-\infty}^{\infty}\sigma^{P_r}\Big|_{P_r=1} \times F_r\rho\,d\phi = \sum_{m=-\infty}^{\infty}\int_0^{2\pi}\sigma^{P_r}\Big|_{P_r=1} \times F_r\rho\,d\phi = \sum_{m=-\infty}^{\infty}\sigma^{F_r*} \\
\sigma^{F_z**} &= \int_0^{2\pi}\sum_{m=-\infty}^{\infty}\sigma^{P_z}\Big|_{P_z=1} \times F_z\rho\,d\phi = \sum_{m=-\infty}^{\infty}\int_0^{2\pi}\sigma^{P_z}\Big|_{P_z=1} \times F_z\rho\,d\phi = \sum_{m=-\infty}^{\infty}\sigma^{F_z*}
\end{aligned}\right\} \quad (3)$$

where $\sigma^{F_{DB}*}$ means the stresses due to a single pair of ring forces and $\sigma^{F_r*}$, $\sigma^{F_z*}$ mean the stresses due to a single ring force.

In calculating the infinite series in eqn (3), it is necessary to obtain the sums of the series of the type $\Sigma_{m=-\infty}^{\infty}\bar{z}^NR^M$ ($N = 0, 1, 2, 3$, $M = -3, -5, -7$). Since they cannot be obtained easily, the following method of calculation is used. First, the range of the summation is divided into three parts as shown in eqn (4).

$$\left.\begin{aligned}
\sum_{m=-\infty}^{\infty}\sigma^{P_{DB}} &= \sum_{m=m_0}^{\infty}\sigma^{P_{DB}} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{P_{DB}} + \sum_{m=-m_0}^{-\infty}\sigma^{P_{DB}} \\
\sum_{m=-\infty}^{\infty}\sigma^{P_r} &= \sum_{m=m_0}^{\infty}\sigma^{P_r} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{P_r} + \sum_{m=-m_0}^{-\infty}\sigma^{P_r} \\
\sum_{m=-\infty}^{\infty}\sigma^{P_z} &= \sum_{m=m_0}^{\infty}\sigma^{P_z} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{P_z} + \sum_{m=-m_0}^{-\infty}\sigma^{P_z}.
\end{aligned}\right\} \quad (4)$$

If we take $m_0$ sufficiently large, $R^{-3}$, $R^{-5}$ and $R^{-7}$ are expanded into binominal series as follows.

$$\left.\begin{aligned}
\sum_{m=m_0}^{\infty} R^{-3} &= \sum_{m=m_0}^{\infty} \{r^2 + \rho^2 - 2r\rho\cos\phi + (z - \zeta - mh)^2\}^{-3/2} \\
&= \sum_{m=m_0}^{\infty} \frac{1}{|z-\zeta-mh|^3}\left\{1 + \frac{r^2+\rho^2-2r\rho\cos\phi}{(z-\zeta-mh)^2}\right\}^{-3/2} \\
&= \sum_{m=m_0}^{\infty} \frac{1}{(\zeta+mh-z)^3}\left\{1 + \frac{3\bar{r}^2}{2(\zeta+mh-z)^2} + \frac{15\bar{r}^4}{8(\zeta+mh-z)^4} \cdots\cdots\right\} \\
&= \frac{1}{h^3}\sum_{m=m_0}^{\infty}\frac{1}{(\bar{\zeta}+m)^3} - \frac{3\bar{r}^2}{2h^5}\sum_{m=m_0}^{\infty}\frac{1}{(\bar{\zeta}+m)^5} + \frac{15\bar{r}^4}{8h^7}\sum_{m=m_0}^{\infty}\frac{1}{(\bar{\zeta}+m)^7}\cdots \\
&= \frac{1}{h^3}\zeta(3,\bar{\zeta}+m_0) - \frac{3\bar{r}^2}{2h^5}\zeta(5,\bar{\zeta}+m_0) + \frac{15\bar{r}^4}{8h^7}\zeta(7,\bar{\zeta}+m_0)\cdots\cdots \\
\sum_{m=m_0}^{\infty} R^{-5} &= \frac{1}{h^5}\zeta(5,\bar{\zeta}+m_0) - \frac{5\bar{r}^2}{2h^7}\zeta(7,\bar{\zeta}+m_0) + \frac{35\bar{r}^4}{8h^9}\zeta(9,\bar{\zeta}+m_0)\cdots\cdots \\
\sum_{m=m_0}^{\infty} R^{-7} &= \frac{1}{h^7}\zeta(7,\bar{\zeta}+m_0) - \frac{7\bar{r}^2}{2h^9}\zeta(9,\bar{\zeta}+m_0) + \frac{63\bar{r}^4}{8h^{11}}\zeta(11,\bar{\zeta}+m_0)\cdots\cdots
\end{aligned}\right\} \quad (5)$$

where

$$\bar{r}^2 = r^2 + \rho^2 - 2r\rho\cos\phi, \quad \bar{\zeta} = (\zeta - z)/h$$

$\zeta(l, x)$ is a Hurwitz zeta function defined in eqn (6).

$$\zeta(l,x) = \sum_{m=0}^{\infty}\frac{1}{(x+m)^l} = \frac{1}{\Gamma(l)}\int_0^{\infty}\frac{t^{l-1}e^{-xt}}{1-e^{-t}}\,dt. \quad (6)$$

The value of $\zeta(l, x)$ can be obtained numerically from eqn (6).

If we take $m_0$ sufficiently large, the total sum of eqn (5) can be obtained accurately by summing up the first few terms of the series. Other series in eqn (3) ($\Sigma_{m=-m_0}^{\infty}\bar{z}^N R^M$, $N = 1, 2, 3$, $M = -3, -5, -7$) can be calculated in a similar manner. When the binominal expansions as shown in eqn (5) are substituted in eqn (3), the integral can be easily obtained in a closed form.

The partial sum of the range ($-m_0 < m < m_0$) in eqn (3) can be calculated by summing up the stress fields of a single ring force as shown in eqn (7). In the case of a single ring force, by using elliptic integrals [6] the integral in eqn (3) can be expressed.

$$\left.\begin{aligned}
\int_0^{2\pi}\sum_{m=-m_0+1}^{m_0-1}\sigma^{P_{DB}}\Big|_{P_{DB}=1}\times F_{DB}\rho\,d\phi &= \sum_{m=-m_0+1}^{m_0-1}\int_0^{2\pi}\sigma^{P_{DB}}\Big|_{P_{DB}=1}\times F_{DB}\rho\,d\phi = \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_{DB}*} \\
\int_0^{2\pi}\sum_{m=-m_0+1}^{m_0-1}\sigma^{P_r}\Big|_{P_r=1}\times F_r\rho\,d\phi &= \sum_{m=-m_0+1}^{m_0-1}\int_0^{2\pi}\sigma^{P_r}\Big|_{P_r=1}\times F_r\rho\,d\phi = \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_r*} \\
\int_0^{2\pi}\sum_{m=-m_0+1}^{m_0-1}\sigma^{P_z}\Big|_{P_z=1}\times F_z\rho\,d\phi &= \sum_{m=-m_0+1}^{m_0-1}\int_0^{2\pi}\sigma^{P_z}\Big|_{P_z=1}\times F_z\rho\,d\phi = \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_z*}
\end{aligned}\right\} \quad (7)$$

The fundamental stress fields $\sigma^{F_{DB}**}$, $\sigma^{F_r**}$, $\sigma^{F_z**}$ can be obtained by the method mentioned above. Finally, they are expressed as follows.

$$\left.\begin{aligned}
\sigma^{F_{DB}**} &= \sum_{m=m_0}^{\infty}\sigma^{F_{DB}*} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_{DB}*} + \sum_{m=-m_0}^{-\infty}\sigma^{F_{DB}*} \\
\sigma^{F_r*} &= \sum_{m=m_0}^{\infty}\sigma^{F_r*} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_r*} + \sum_{m=-m_0}^{-\infty}\sigma^{F_r*} \\
\sigma^{F_z*} &= \sum_{m=m_0}^{\infty}\sigma^{F_z*} + \sum_{m=-m_0+1}^{m_0-1}\sigma^{F_z*} + \sum_{m=-m_0}^{-\infty}\sigma^{F_z*}.
\end{aligned}\right\} \quad (8)$$

The stress fields due to a single ring force $\sigma^{F_{DB}*}$, $\sigma^{F_r*}$, $\sigma^{F_z*}$ are given as follows [6].

$$\left.\begin{aligned}
\sigma_r^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2 r_m^3}\left[2(1-\nu)I_0 + \frac{3(1-2\nu)}{r_m^2}\rho^2(-J_0+J_2) + \frac{15\bar{z}^2}{r_m^4}(-r^2L_0 + 2r\rho L_1 - \rho^2 L_2)\right]\\
\sigma_z^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2 r_m^3}\left[I_0 + \frac{6\bar{z}^2}{r_m^2}J_0 + \frac{15\bar{z}^4}{r_m^4}(-L_0)\right]\\
\tau_{rz}^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2 r_m^3}\left[\frac{3\bar{z}}{r_m^2}(rJ_0 - \rho J_1) + \frac{15\bar{z}^3}{r_m^4}(-rL_0 + \rho L_1)\right]\\
\sigma_r^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)r_m^3}\Big[(1-2\nu)(-\rho I_0 - rI_1 + 2\rho I_2)\\
&\quad + \frac{3}{r_m^2}\{r^2\rho J_0 - r(r^2+2\rho^2)J_1 + \rho(2r^2+\rho^2)J_2 - r\rho^2 J_3\}\Big]\\
\sigma_z^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)r_m^3}\left[(1-2\nu)(-\rho I_0 + rI_1) + \frac{3\bar{z}^2}{r_m^2}(\rho J_0 - rJ_1)\right]\\
\tau_{rz}^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)r_m^3}\left[(1-2\nu)\bar{z}(-I_1) + \frac{3\bar{z}}{r_m^2}\{r\rho J_0 - (r^2+\rho^2)J_1 + r\rho J_2\}\right]\\
\sigma_r^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)r_m^3}\left[(1-2\nu)\bar{z}I_0 + \frac{3\bar{z}}{r_m^2}(-r^2J_0 + 2r\rho J_1 - \rho^2 J_2)\right]\\
\sigma_z^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)r_m^3}\left[(1-2\nu)\bar{z}(-I_0) + \frac{3\bar{z}^3}{r_m^2}(-J_0)\right]\\
\tau_{rz}^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)r_m^3}\left[(1-2\nu)(-rI_0 + \rho I_1) + \frac{3\bar{z}^2}{r_m^2}(-rJ_0 + \rho J_1)\right]
\end{aligned}\right\}\quad (9a)$$

where

$$\left.\begin{aligned}
&r_m = \sqrt{2r\rho}, \quad \bar{z} = z - \zeta,\\
&I_n = \int_0^\pi \frac{\cos^n\phi}{(e-\cos\phi)^{3/2}}\,\mathrm{d}\phi, \quad J_n = \int_0^\pi \frac{\cos^n\phi}{(e-\cos\phi)^{5/2}}\,\mathrm{d}\phi, \quad L_n = \int_0^\pi \frac{\cos^n\phi}{(e-\cos\phi)^{7/2}}\,\mathrm{d}\phi\\
&I_0 = \frac{1}{e^2-1}K_1,\\
&I_1 = \frac{e}{e^2-1}K_1 - K_2,\\
&I_2 = \frac{2e^2-1}{e^2-1}K_1 - 2eK_2,\\
&J_0 = \frac{4e}{3(e^2-1)^2}K_1 - \frac{1}{3(e^2-1)}K_2,\\
&J_1 = \frac{e^2+3}{3(e^2-1)^2}K_1 - \frac{e}{3(e^2-1)}K_2,\\
&J_2 = -\frac{2e(e^2-3)}{3(e^2-1)^2}K_1 + \frac{2e^2-3}{3(e^2-1)}K_2,\\
&J_3 = \frac{-8e^4+15e^2-3}{3(e^2-1)^2}K_1 + \frac{e(8e^2-9)}{3(e^2-1)}K_2,\\
&L_0 = \frac{23e^2+9}{15(e^2-1)^3}K_1 - \frac{8e}{15(e^2-1)^2}K_2,
\end{aligned}\right\}\quad (9b)$$

$$L_1 = \frac{e(3e^2+29)}{15(e^2-1)^3} K_1 - \frac{3e^2+5}{15(e^2-1)^2} K_2,$$

$$L_2 = \frac{-2e^4+19e^2+15}{15(e^2-1)^3} K_1 + \frac{2e(e^2-5)}{15(e^2-1)^2} K_2,$$

$$K_1 = \int_0^\pi (e-\cos\phi)^{1/2}\,\mathrm{d}\phi = \frac{2\sqrt{2}}{k} E(k)$$

$$K_2 = \int_0^\pi (e-\cos\phi)^{-1/2}\,\mathrm{d}\phi = \sqrt{2}\,kK(k).$$

The complete elliptic integrals

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\lambda}{\sqrt{1-k^2\sin^2\lambda}}, \quad E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\lambda}\,\mathrm{d}\lambda \tag{9b}$$

have the argument

$$k = \sqrt{\frac{2}{e+1}}, \quad e = 1 + \frac{(r-e)^2+(z-\zeta)^2}{2r\rho}.$$

In calculating the second terms in eqn (8) ($\Sigma_{m=-m_0+1}^{m_0-1}\sigma^{F_{DB}*}$, $\Sigma_{m=-m_0+1}^{m_0-1}\sigma^{F_r*}$, $\Sigma_{m=-m_0+1}^{m_0-1}\sigma^{F_z*}$), eqn (9) is used directly. The first terms in eqn (8) can be expressed as follows.

$$\begin{aligned}
\sum_{m=m_0}^{\infty} \sigma_r^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2}\Bigg[2(1-\nu)\sum_{l=0}^{\infty} S_1(l)\zeta_m(2l+3)C_0(l) \\
&\quad + 3(1-2\nu)\rho^2\sum_{l=0}^{\infty} S_2(l)\zeta_m(2l+5)\{-C_0(l)+C_2(l)\} \\
&\quad + 15\sum_{l=0}^{\infty} S_3(l)\zeta_m(2l+5)\{-r^2C_0(l)+2r\rho C_1(l)-\rho^2C_2(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty} \sigma_z^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2}\Bigg[\sum_{l=0}^{\infty} S_1(l)\zeta_m(2l+3)C_0(l) \\
&\quad + 6\sum_{l=0}^{\infty} S_2(l)\zeta_m(2l+3)C_0(l) + 15\sum_{l=0}^{\infty} S_3(l)\zeta_m(2l+3)\{-C_0(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty} \tau_{rz}^{F_{DB}*} &= \frac{F_{DB}\rho(1-2\nu)}{4\pi(1-\nu)^2}\Bigg[3\sum_{l=0}^{\infty} S_2(l)\zeta_m(2l+4)\{rC_0(l)-\rho C_1(l)\} \\
&\quad + 15\sum_{l=0}^{\infty} S_3(l)\zeta_m(2l+4)\{-rC_0(l)+\rho C_1(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty} \sigma_r^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty} S_1(l)\zeta_m(2l+3) \\
&\quad \times\{-\rho C_0(l)-rC_1(l)+2\rho C_2(l)\} + 3\sum_{l=0}^{\infty} S_2(l)\zeta_m(2l+5) \\
&\quad \times\{r^2\rho C_0(l)-r(r^2+2\rho^2)C_1(l)+\rho(2r^2+\rho^2)C_2(l)-r\rho^2C_3(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty} \sigma_z^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty} S_1(l)\zeta_m(2l+3)\{-\rho C_0(l)+rC_1(l)\} \\
&\quad + 3\sum_{l=0}^{\infty} S_2(l)\zeta_m(2l+3)\{\rho C_0(l)-rC_1(l)\}\Bigg]
\end{aligned} \tag{10a}$$

$$
\left.
\begin{aligned}
\sum_{m=m_0}^{\infty}\tau_{rz}^{F_r*} &= \frac{F_r\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty}S_1(l)\zeta_m(2l+2)\{-C_1(l)\} \\
&\quad + 3\sum_{l=0}^{\infty}S_2(l)\zeta_m(2l+4)\{r\rho C_0(l)-(r^2+\rho^2)C_1(l)+r\rho C_2(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty}\sigma_r^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty}S_1(l)\zeta_m(2l+2)C_0(l) \\
&\quad + 3\sum_{l=0}^{\infty}S_2(l)\zeta_m(2l+4)\{-r^2C_0(l)+2r\rho C_1(l)-\rho^2C_2(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty}\sigma_z^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty}S_1(l)\zeta_m(2l+2)\{-C_0(l)\}+3\sum_{l=0}^{\infty}S_2(l)\zeta_m(2l+2)\{-C_0(l)\}\Bigg] \\
\sum_{m=m_0}^{\infty}\tau_{rz}^{F_z*} &= \frac{F_z\rho}{4\pi(1-\nu)}\Bigg[(1-2\nu)\sum_{l=0}^{\infty}S_1(l)\zeta_m(2l+3)\{-rC_0(l)+\rho C_1(l)\} \\
&\quad + 3\sum_{l=0}^{\infty}S_2(l)\zeta_m(2l+3)\{-rC_0(l)+\rho C_1(l)\}\Bigg]
\end{aligned}
\right\}
$$

where

$$
\left.
\begin{aligned}
S_1(l) &= (-1)^l\frac{(2l+1)!}{l!2^l} \\
S_2(l) &= (-1)^l\frac{(2l+3)!}{3l!2^l} \\
S_3(l) &= (-1)^l\frac{(2l+5)!}{15l!2^l}
\end{aligned}
\right\}
\tag{10b}
$$

$$
C_n(l) = \int_0^{\pi}\cos^n\phi(r^2+\rho^2-2r\rho\cos\phi)^l\,\mathrm{d}\phi \tag{10c}
$$

$$
\left.
\begin{aligned}
\zeta_m(l) &= (-1)^{l+1}\zeta(l,\bar{\zeta}+m_0)/h^l \\
\bar{\zeta}\pm &= (\zeta - z)/h^l
\end{aligned}
\right\}
\tag{10d}
$$

$\zeta(l, x)$ is defined in eqn (6).

The third terms in eqn (8) ($\Sigma_{m=-m_0}^{-\infty}\sigma^{F_{DB}*}$, $\Sigma_{m=-m_0}^{-\infty}\sigma^{F_r*}$, $\Sigma_{m=-m_0}^{-\infty}\sigma^{F_z*}$) can be obtained by replacing eqn (10d) with eqn (10e).

$$
\left.
\begin{aligned}
\zeta_m(l) &= \zeta(l,\bar{\zeta}+m_0)/h^l \\
\bar{\zeta} &= (z-\zeta)/h
\end{aligned}
\right\}
\tag{10e}
$$

## 4. PROCEDURE FOR NUMERICAL SOLUTIONS

Figure 5 shows imaginary boundaries where body force or a pair of body forces are distributed. A pair of body forces is applied along the part $A'B$ in addition to the part $AB$ which should become a circumferential crack, because it makes the shear stress $\tau_{rz}$ at $B$ small and consequently the boundary conditions can be satisfied easily. From the symmetry of the problem, the boundary conditions have only to be satisfied along the part $ABC$ by using the fundamental solutions given in eqns (8)–(10).

It is difficult to determine in closed forms the body force densities which satisfy the boundary conditions completely. Therefore, the imaginary boundaries are divided and the problem is solved numerically. The imaginary boundary of the circumferential crack is divided into $n_1$ equal intervals and the imaginary cylindrical surface into $n_2$ intervals. The densities of body forces (or a pair of

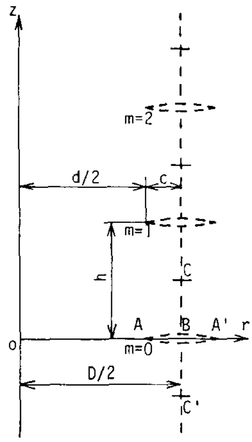


Fig. 5. Imaginary crack surface and imaginary cylindrical surface in an infinite body.

body forces), which are assumed to be constant in each interval, are determined from the boundary condition at the midpoint of each interval.

The influence coefficients $\sigma_i^{\rho_{DB}}$, $\sigma_i^{\rho_r}$, $\sigma_i^{\rho_z}$, which mean the stresses induced at the midpoint of $i$th interval by the unit body force acting at the $j$th interval can be written as eqn (11).

$$\left.\begin{array}{l}
\sigma_i^{\rho_{DB}j} = \displaystyle\int_j \sigma^{F_{DB}**}\Big|_{F_{DB}=1} \frac{(1-\nu)^2}{1-2\nu} 4\sqrt{C^2-(\rho-D/2)^2}\,\mathrm{d}\rho \;(j = 1 \sim n_1) \\
\left.\begin{array}{l}
\sigma_i^{\rho_r j} = \displaystyle\int_j \sigma^{F_r**}\Big|_{F_r=1} \mathrm{d}\zeta \\
\sigma_i^{\rho_z j} = \displaystyle\int_j \sigma^{F_z**}\Big|_{F_z=1} \mathrm{d}\zeta
\end{array}\right\} (j = n_1+1 \sim n_1+n_2)
\end{array}\right\} \quad (11)$$

where $\int_j$ stands for integration of the $j$th interval. The integrations in eqn (11) are performed numerically using Gauss's formula. The boundary conditions at the midpoint of the $i$th interval are expressed by using the influence coefficients as follows.

$$\left.\begin{array}{l}
\displaystyle\sum_{j=1}^{n_1} \rho_{DBj}\sigma_z^{\rho_{DB}j} + \sum_{j=n_1+1}^{n_1+n_2} (\rho_{rj}\sigma_z^{\rho_r j} + \rho_{zj}\sigma_z^{\rho_z j}) + \sigma_z^\infty = 0 \qquad (i = 1 \sim n_1) \\
\left.\begin{array}{l}
\displaystyle\sum_{j=1}^{n_1} \rho_{DBj}\sigma_r^{\rho_{DB}j} + \sum_{j=n_1+1}^{n_1+n_2} (\rho_{rj}\sigma_r^{\rho_r j} + \rho_{zj}\sigma_r^{\rho_z j}) = 0 \\
\displaystyle\sum_{j=1}^{n_1} \rho_{DBj}\tau_{rz}^{\rho_{DB}j} + \sum_{j=n_1+1}^{n_1+n_2} (\rho_{rj}\tau_{rz}^{\rho_r j} + \rho_{zj}\tau_{rz}^{\rho_z j}) = 0
\end{array}\right\} \quad (i = n_1+1 \sim n_1+n_2)
\end{array}\right\} \quad (12)$$

where $\sigma_z^\infty$ is the nominal stress for the cylindrical diameter $D$. The first, second and third equations in eqn (12) correspond to $\sigma_z = 0$ at the crack surface, $\sigma_r = 0$ at the cylindrical surface, and $\tau_{rz} = 0$ at the cylindrical surface, respectively.

The body force densities are determined by solving the $(n_1 + 2n_2)$ linear simultaneous equations (12). The dimensionless stress intensity factor $F_I$ is obtained from eqn (13).

$$F_I = \rho_{DB_1} = \frac{K_I}{\sigma_z^\infty \sqrt{\pi c}} \quad (13)$$

where $K_I$ is the stress intensity factor, and $\rho_{DB_1}$ is the density of a pair of body forces at the first ($j = 1$) interval.

Table 1. Dimensionless stress intensity factors $F_I$ for a single circumferential crack (Fig. 1); $K_I = F_I\sigma_z^\infty\sqrt{\pi c}$ $\nu = 0.3$

| 2c/D | Present analysis | Keer et al. | Atsumi Shindo | Benthem Koiter | Beuckner | Deep notch |
|---|---|---|---|---|---|---|
| 0.02 | 1.133 | – | – | 1.125 | – | 3.644 |
| 0.03 | 1.139 | – | – | 1.127 | – | 3.022 |
| 0.05 | 1.150 | 1.1513 | 1.152 | 1.133 | – | 2.415 |
| 0.1 | 1.180 | 1.1807 | 1.181 | 1.153 | – | 1.852 |
| 0.2 | 1.261 | 1.2608 | 1.261 | 1.225 | 1.240 | 1.563 |
| 0.3 | 1.393 | 1.3904 | 1.390 | 1.353 | 1.365 | 1.559 |
| 1/3 | 1.452 | – | – | 1.412 | – | 1.591 |
| 0.4 | 1.602 | 1.597 | 1.596 | 1.561 | 1.584 | 1.701 |
| 0.5 | 1.940 | 1.932 | 1.928 | 1.901 | 1.921 | 2.000 |
| 0.6 | 2.516 | 2.502 | 2.494 | 2.481 | – | 2.552 |
| 2/3 | 3.158 | – | – | 3.128 | – | 3.182 |
| 0.7 | 3.618 | 3.598 | 3.571 | 3.590 | – | 3.637 |
| 0.8 | 6.243 | 6.201 | 6.115 | 6.223 | – | 6.250 |
| 0.9 | 16.67 | 16.46 | – | 16.66 | – | 16.67 |

Since the error due to the finiteness of division number $n_t(=n_1+n_2)$ is nearly proportional to $1/n_t$ [5, 6], the values of stress intensity factor corresponding to $n_t \to \infty$ is obtained by extrapolation of the two values of $F_I$ corresponding to two values of $n_t$. Poisson's ratio is assumed to be 0.3.

## 5. RESULTS AND DISCUSSION

### 5.1 *Stress intensity factor of a single circumferential crack*

In Table 1, the stress intensity factors of a single circumferential crack are compared with the results of Keer–Freedmann–Watts [3], the results of Atsumi–Shindo [4], an approximate formula by Benthem–Koiter [2], the results of Beuckner [1], and the solution of a deep circumferential crack.

Formula by Benthem–Koiter [2] is:

$$K_I=\frac{1}{2}\sigma_{\rm net}\sqrt{\pi dc/D}\left\{1+\frac{1}{2}\left(\frac{d}{D}\right)+\frac{3}{8}\left(\frac{d}{D}\right)^2-0.363\left(\frac{d}{D}\right)^3+0.731\left(\frac{d}{D}\right)^4\right\}. \tag{14}$$

Solution of a deep circumferential crack is:

$$K_I=\frac{1}{2}\sigma_{\rm net}\sqrt{\pi d/2} \tag{15}$$

where $\sigma_{\rm net}$ is the nominal stress for the minimum diameter $d[\sigma_{\rm net}=\sigma_z^\infty(D/d)^2]$.

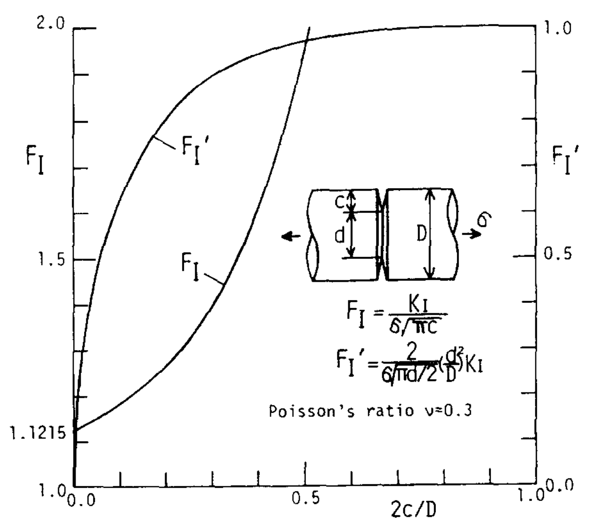


Fig. 6. Stress intensity factors of a single circumferential crack; $K_I = F_I\sigma_z^\infty\sqrt{\pi c}$, $K_I = F_I'\frac{1}{2}\sigma_{\rm net}\sqrt{\pi(d/2)}$.

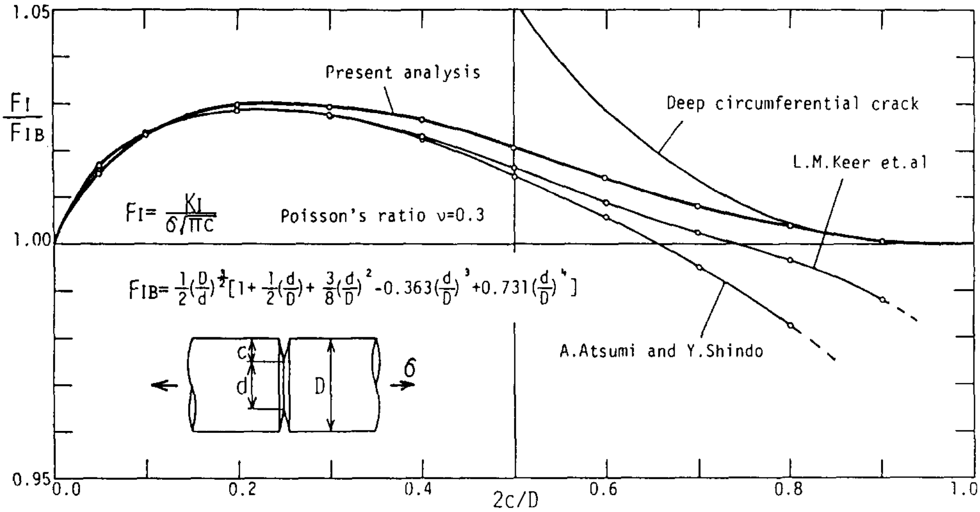


Fig. 7. Relation between $F_I/F_{IB}$ and $\lambda = 2c/D$ ($F_{IB}$:$F_I$ values of an approximate formula by Benthem and Koiter).

Figure 6 shows two kinds of dimensionless stress intensity factors $F_I$, $F_I'$.

$$F_I = K_I/(\sigma_z^\infty\sqrt{\pi c}), \quad F_I' = K_I/\{\tfrac{1}{2}\sigma_{\text{net}}\sqrt{\pi d/2}\}. \tag{16}$$

The results of the present analysis agree with the exact values in the two limiting cases, $2c/D \to 0$ and $2c/D \to 1$.

As shown in Table 1, the results of other researchers are in good agreement with the present results except the case of deep crack. In order to examine the accuracy of them more strictly, the results in Table 1 are plotted in Fig. 7. In Fig. 7, the ordinate represents the ratio $F_I/F_{IB}$, where $F_{IB}$ denotes $F_I$ values of an approximate formula by Benthem–Koiter, and the abscissa represents the relative crack depth $2c/D$. As $2c/D \to 0$ or $2c/D \to 1$, the ratio $F_I/F_{IB}$ must approach unity because the formula by Benthem–Koiter gives the exact values in the two limiting cases. The results of the present analysis seems to be more reliable than the results of other papers in the case of deep crack. In the case of shallow crack, the results of Keer *et al.* and the results of Atsumi–Shindo are in good agreement with the present results. The approximate formula by Benthem–Koiter has a non-conservative error about 3%.

### 5.2 *Stress intensity factor of an infinite row of circumferential crack*

In Table 2, the stress intensity factors of a row of circumferential cracks are shown. In order to investigate the interference effect of a row of circumferential cracks, the results in Table 2 are plotted in Fig. 8.

In Fig. 8, the ordinate represents $F_I/F_{I0}$, where $F_{I0}$ denotes $F_I$ values for a single circumferential crack, and the abscissa represents the relative crack depth $2c/D$. As $2c/D \to 0$, the stress intensity factor for a row of circumferential cracks approaches smoothly the value corresponding to that for a row of edge cracks in a semi-infinite plate under tension [9]. As the cracks become deep, the stress intensity factor approaches the corresponding value for a single deep circumferential crack. That is, the interference effect among cracks disappears as the relative crack depth $2c/D$ tends to unity.

Table 2. Dimensionless stress intensity factors $F_I$ for an infinite row of circumferential cracks (Fig. 2); $K_I = F_I\sigma_z^\infty\sqrt{\pi c}$ $\nu = 0.3$

| 2c/D \ c/h | $F_I$ | | | | | $F_I/F_I\|_{c/h\to 0}$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0 | 0.2 | 1/3 | 0.5 | 2/3 | 0.0 | 0.2 | 1/3 | 0.5 | 2/3 |
| 0.0 | 1.122 | 1.261 | 1.452 | 1.940 | 3.158 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.872 | 1.176 | 1.439 | 1.942 | – | 0.778 | 0.933 | 0.991 | 1.001 | – |
| 0.3 | 0.726 | 1.027 | 1.335 | 1.917 | – | 0.647 | 0.814 | 0.919 | 0.988 | – |
| 0.4 | 0.625 | 0.906 | 1.212 | 1.839 | – | 0.558 | 0.719 | 0.835 | 0.948 | – |
| 0.5 | 0.558 | 0.821 | 1.113 | 1.746 | 3.092 | 0.498 | 0.651 | 0.767 | 0.900 | 0.979 |

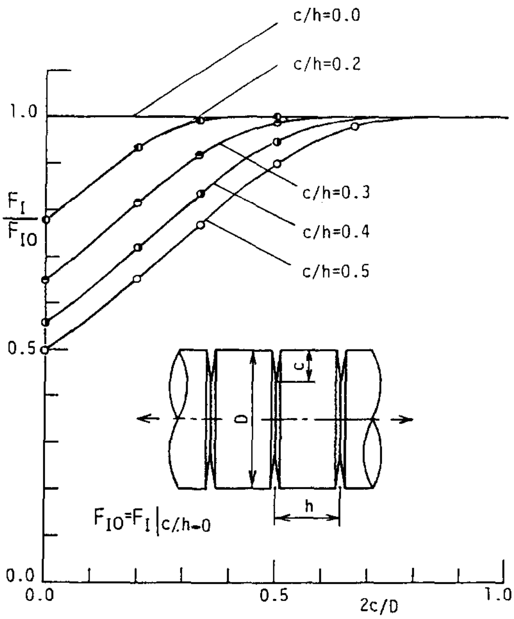


Fig. 8. Interference effect of an infinite row of circumferential crack; $K_I = F_I\sigma_z^\infty\sqrt{\pi c}$. $\nu = 0.3$

## 6. CONCLUSION

In this paper, the crack problems of a cylindrical bar having a circumferential crack and a cylindrical bar having an infinite row of circumferential cracks under tension are analyzed by the body force method. On the basis of the numerical results which have been presented, the following conclusions can be made.

(1) The stress intensity factors of a single circumferential crack obtained by the present analysis are in good agreement with the results of Keer *et al* [3] and the results of Atsumi–Shindo [4], especially for the cases of shallow crack. As the crack becomes very deep, the present results approach the exact value of a deep circumferential crack. An approximate formula by Benthem–Koiter [2] has a non-conservative error about 3%.

(2) As the crack becomes very shallow, the stress intensity factor of an infinite row of circumferential cracks approaches the value corresponding to that of a row of edge cracks in a semi-infinite plate under tension. As the crack becomes very deep, it approaches the values corresponding to that of a single deep circumferential crack.

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