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Stress intensity factors in two bonded elastic layers with a single edge crack under various loading conditions

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Abstract. In this paper, the plane problems of a single edge crack in two bonded elastic layers and in an elastic surface layer bonded to an elastic semi-infinite plane are analyzed by the body force method. The stress fields induced by a point force and a displacement discontinuity in two bonded elastic half-planes obtained by Hetenyi's solution, in other words, Green's function in closed forms, are used as fundamental solutions to solve those problems. The boundary conditions for stress free-edges of the layers and the crack surface are satisfied by superposing the distributed fundamental solutions and adjusting their densities. The stress intensity factors are systematically calculated for the various geometrical conditions and the various stiffness ratios of the layers.

1. Introduction

In recent years, composite materials have been widely used for high performance structures where both light weight and high strength are pursued. In general composite materials have many interfaces. However, the plane problems of a crack in two bonded elastic layers are basically important to understand the fatigue crack extension behavior near the interface. Therefore, there has been considerable research into the crack near the interface in an elastic infinite plane [1–6]; however, the exact analysis of the finite plate having the interface has been expected to promote more advanced experimental research into the fracture of the composite materials.

In previous work, Yu-uki et al. [7] have applied Hetenyi's solution to the body force method (BFM), and have analyzed the stress intensity factor for an imbedded crack crossing the interface of the two bonded elastic layers. Lu and Erdogan [8] have analyzed cracks perpendicular to and on the interface in two bonded elastic layers formulating the problem in terms of a coupled system of four singular integral equations. However, generally the integral kernel to formulate the problem is very complicated and the expression may have some mistakes [8, 9]. Lately the boundary element method (BEM) and the finite element method (FEM) have been used to calculate the stress intensity factor of the problem [10, 11]. However, in the fatigue experimental test the exact analysis has been required to study the fracture of layered materials.

In this paper, the stress intensity factor in two bonded elastic layers with a single edge crack under various remote loading conditions (Fig. 1) are analyzed by the BFM. As the fundamental solutions, the stress fields induced by a point force and a displacement discontinuity in an infinite plate having the interface, which are obtained by Hetenyi's solution [12], are used. In the analysis, we especially consider the resultant P and the bending moment M_p which appear at the mid-point of the ligament of the two bonded layers in order to get the accurate solution for the problem of a single edge crack [13]. In addition, the problem of a cracked elastic surface layer bonded to an elastic half space and the problem of a crack crossing the interface are also analyzed and the results are compared with previous research.

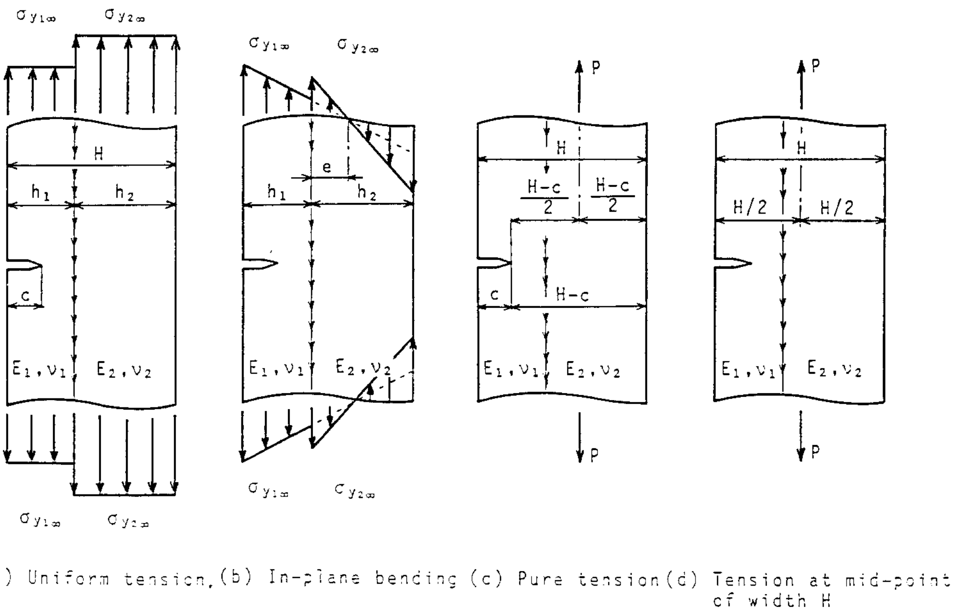


Fig. 1. Two bonded elastic layers with a single edge crack.

2. Fundamental solutions

In this analysis, the stress fields induced by a point force and a displacement discontinuity shown in Figs. 2(a, b), which are obtained by Hetenyi's solutions [12] are used as fundamental solutions. The point forces and the displacement discontinuities are distributed continuously along the prospective sites of the stress free boundaries for edges and crack respectively.

2.1. Stress fields induced by a point force in an infinite plate having the interface

When a point force P or Q acting in the x or y-direction is applied to a point (ξ, η) in material 1, the stresses at (x, y) are given by (1)–(12).

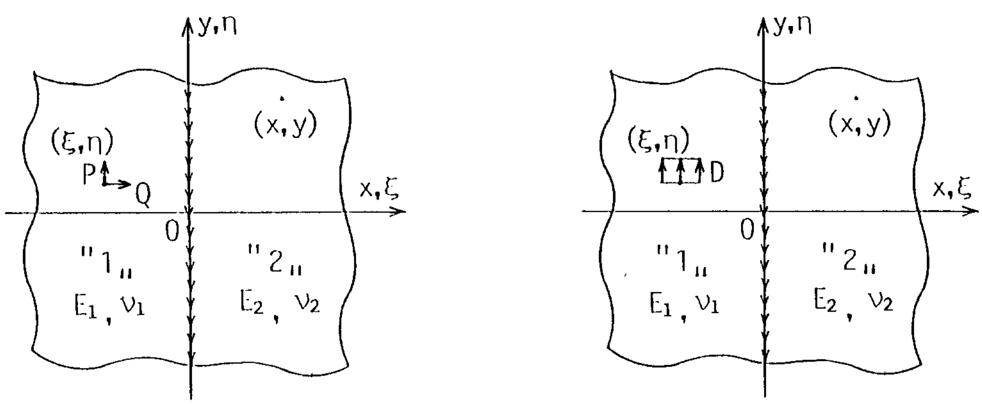


Fig. 2(a). Fundamental stress field for the point force of an infinite plane having the interface.
(*b*). Stress field for the displacement discontinuity of an infinite plane having the interface.

(A) In the case that a point (x, y) exists in material 1

$$\sigma_{x1}^{P} = \frac{P(y-\eta)}{2\pi(\kappa_1+1)}\left[\frac{\kappa_1-1}{r_1^2} - \frac{4(x-\xi)^2}{r_1^4} - \frac{A\kappa_1 - B}{r_2^2} + 4A\frac{2\xi^2 + (\kappa_1-1)(x+\xi)\xi - \kappa_1(x+\xi)^2}{r_2^4} + 32A\frac{(x+\xi)^2 x\xi}{r_2^6}\right], \tag{1}$$

$$\sigma_{y1}^{P} = \frac{P(y-\eta)}{2\pi(\kappa_1+1)}\left[-\frac{\kappa_1-3}{r_1^2} - \frac{4(x-\xi)^2}{r_1^4} - \frac{3A\kappa_1 + B}{r_2^2} - 4A\frac{2\xi^2 - (5-\kappa_1)(x+\xi)\xi - \kappa_1(x+\xi)^2}{r_2^4} - 32A\frac{x\xi(x+\xi)^2}{r_2^6}\right], \tag{2}$$

$$\tau_{xy1}^{P} = \frac{P}{2\pi(\kappa_1+1)}\left[-\frac{(\kappa_1+3)(x-\xi)}{r_1^2} + 4\frac{(x-\xi)^3}{r_1^4} + \frac{2A(\kappa_1-1)\xi - (3A\kappa_1+B)(x+\xi)}{r_2^2} + 4A(x+\xi)\frac{-6\xi^2 + (7-\kappa_1)(x+\xi)\xi + \kappa_1(x+\xi)^2}{r_2^4} - 32A\frac{(x+\xi)^3 x\xi}{r_2^6}\right], \tag{3}$$

$$\sigma_{x1}^{Q} = \frac{Q}{2\pi(\kappa_1+1)}\left[-\frac{(\kappa_1-1)(x-\xi)}{r_1^2} - \frac{4(x-\xi)^3}{r_1^4} + \frac{-2A(\kappa_1-1)\xi + (A\kappa_1 - B)(x+\xi)}{r_2^2} + 4A(x+\xi)\frac{-6\xi^2 + (\kappa_1+5)(x+\xi)\xi - \kappa_1(x+\xi)^2}{r_2^4} - 32A\frac{(x+\xi)^3 x\xi}{r_2^6}\right], \tag{4}$$

$$\sigma_{y1}^{Q} = \frac{Q}{2\pi(\kappa_1+1)}\left[\frac{(\kappa_1-1)(x-\xi)}{r_1^2} - \frac{4(x-\xi)(y-\eta)^2}{r_1^4} - \frac{2A(\kappa_1-1)\xi + (A\kappa_1 - B)(x+\xi)}{r_2^2} - 4A\frac{2(x+\xi)\xi^2 - (\kappa_1-7)(y-\eta)^2\xi + \kappa_1(x+\xi)(y-\eta)^2}{r_2^4} + 32A(y-\eta)^2\xi\frac{(x+\xi)\xi + (y-\eta)^2}{r_2^6}\right], \tag{5}$$

$$\tau_{xy1}^{Q} = \frac{Q(y-\eta)}{2\pi(\kappa_1+1)}\left[-\frac{\kappa_1+3}{r_1^2} + \frac{4(y-\eta)^2}{r_1^4} - \frac{3A\kappa_1 + B}{r_2^2} + 4A\frac{6\xi^2 + (\kappa_1-5)(x+\xi)\xi + \kappa_1(y-\eta)^2}{r_2^4} + 32A(y-\eta)^2\frac{x\xi}{r_2^6}\right]. \tag{6}$$

(B) In the case that a point (x, y) exists in material 2

$$\sigma_{y2}^{P} = \frac{P(y-\eta)}{2\pi(\kappa_1+1)}\left[\frac{(1-A)\kappa_1 - (1-B)}{r_1^2} + 4(x-\xi)\frac{(B-A)\xi - (1-B)(x-\xi)}{r_1^4}\right], \tag{7}$$

$$\sigma_{x2}^{P} = \frac{P(y-\eta)}{2\pi(\kappa_1+1)}\left[\frac{-(1-A)\kappa_1 - 3(1-B)}{r_1^2} - 4(x-\xi)\frac{(B-A)\xi - (1-B)(x-\xi)}{r_1^4}\right], \tag{8}$$

$$\tau_{xy2}^{P} = \frac{P}{2\pi(\kappa_1+1)}\left[\frac{2(B-A)\xi - (\kappa_1 - A\kappa_1 + 3 - 3B)(x-\xi)}{r_1^2}\right.$$
$$\left. - 4(x-\xi)^2\frac{(B-A)\xi - (1-B)(x-\xi)}{r_1^4}\right], \tag{9}$$

$$\sigma_{x2}^{Q} = \frac{Q}{2\pi(\kappa_1+1)}\left[\frac{-2(B-A)\xi - \{(1-A)\kappa_1 - (1-B)\}(x-\xi)}{r_1^2}\right.$$
$$\left. + 4(x-\xi)^2\frac{(B-A)\xi - (1-B)(x-\xi)}{r_1^4}\right], \tag{10}$$

$$\sigma_{y2}^{Q} = \frac{Q}{2\pi(\kappa_1+1)}\left[\frac{-2(B-A)\xi + \{(1-A)\kappa_1 - (1-B)\}(x-\xi)}{r_1^2}\right.$$
$$\left. + 4(y-\eta)^2\frac{(B-A)\xi - (1-B)(x-\xi)}{r_1^4}\right], \tag{11}$$

$$\tau_{xy2}^{Q} = \frac{Q(y-\eta)}{2\pi(\kappa_1+1)}\left[\frac{-(1-A)\kappa_1 - 3(1-B)}{r_1^2} + 4\frac{(B-A)(x-\xi)\xi + (1-B)(y-\eta)^2}{r_1^4}\right]. \tag{12}$$

2.2. Stress fields induced by a displacement discontinuity with the magnitude D in an infinite plate having the interface

When a displacement discontinuity acting in the y-direction is applied to a point (ξ, η) in material 1, the stresses at (x, y) are given by (13)–(18).

(A) In the case that a point (x, y) exists in material 1

$$\sigma_{x1}^{D} = \frac{DG_1}{2\pi(\kappa_1-1)}\left[-\frac{2(\kappa_1-1)^2}{(1+\kappa_1)r_1^2} + 2\frac{(\kappa_1^2-1)y_1^2 + x_1^2\{\kappa_1(\kappa_1-8)+23\}}{(1+\kappa_1)r_1^4}\right.$$
$$- 16x_1\frac{y_1^2(1+\kappa_1) + (3-\kappa_1)x_1^2}{(1+\kappa_1)r_1^6} + 2\frac{A(\kappa_1^2 - 2\kappa_1 + 3) - 2B}{(1+\kappa_1)r_2^2}$$
$$- 2\frac{(A\kappa_1 - B)(1+\kappa_1)y_1^2 + 2A(1+\kappa_1)\{2\xi^2 + (\kappa_1-1)x_2\xi - \kappa_1 x_2^2\}}{(1+\kappa_1)r_2^4}$$
$$\frac{+(3-\kappa_1)[x_2\{A\kappa_1(5\xi+7x) + 2A\xi - Bx_2\} + 12A\xi(2x+3\xi)}{(1+\kappa_1)r_2^4}$$
$$\frac{-2A(\kappa_1+5)(x+3\xi)x_2]}{(1+\kappa_1)r_2^4} + 16A\frac{(1+\kappa_1)\{2\xi^2 + (\kappa_1-1)x_2\xi - \kappa_1 x_2^2\}y_1^2}{(1+\kappa_1)r_2^6}$$
$$\frac{-2(1+\kappa_1)x_2^2 x\xi + (3-\kappa_1)[x_2^2\{6\xi^2 - (\kappa_1+5)x_2\xi + \kappa_1 x_2^2\}}{(1+\kappa_1)r_2^6}$$

$$\frac{-2x\{x(x+3\xi)^2+4\xi^3\}]}{(1+\kappa_1)r_2^6}+192Ax_2^2x\xi\frac{(1+\kappa_1)y_1^2+(3-\kappa_1)x_2^2}{(1+\kappa_1)r_2^8}\Bigg], \tag{13}$$

$$\sigma_{y1}^D=\frac{DG_1}{2\pi(\kappa_1-1)}\Bigg[\frac{2(3+\kappa_1^2)}{(1+\kappa_1)r_1^2}-2\frac{(\kappa_1^2-2\kappa_1+5)x_1^2+(\kappa_1^2+6\kappa_1-3)y_1^2}{(1+\kappa_1)r_1^4}$$

$$+32x_1^2y_1\frac{(\kappa_1-1)}{(1+\kappa_1)r_1^6}+2\frac{A\kappa_1(3\kappa_1-4)+2B+3A}{(1+\kappa_1)r_2^2}$$

$$+2\frac{(1+\kappa_1)[2A\{2\xi^2-(5-\kappa_1)x_2\xi-\kappa_1x_2^2\}-(3A\kappa_1+B)y_1^2]}{(1+\kappa_1)r_2^4}$$

$$\frac{+(3-\kappa_1)[x_2\{2A(\kappa_1-1)\xi+(A\kappa_1-B)x_2\}-2A\{\xi(4x+6\xi)+7y_1^2\}]}{(1+\kappa_1)r_2^4}$$

$$+16A\frac{(1+\kappa_1)[2x\xi x_2^2-y_1^2\{2\xi^2-(5-\kappa_1)x_2\xi-\kappa_1x_2^2\}]}{(1+\kappa_1)r_2^6}$$

$$\frac{+(3-\kappa_1)[2y_1^2\{\xi(2x+3\xi)+y_1^2\}+x_2\{2x_2\xi^2+(7\xi+\kappa_1x)y_1^2\}]}{(1+\kappa_1)r_2^6}$$

$$-192Ax_2y_1^2\frac{(1+\kappa_1)x_2x+(3-\kappa_1)(x_2\xi+y_1^2)}{(1+\kappa_1)r_2^8}\Bigg], \tag{14}$$

$$\tau_{xy1}^D=\frac{DG_1y_1}{2\pi(\kappa_1-1)}\Bigg[-\frac{8(\kappa_1+3)x_1}{(1+\kappa_1)r_1^4}+16x_1\frac{(\kappa_1+1)x_1^2+(3-\kappa_1)y_1^2}{(1+\kappa_1)r_1^6}$$

$$+2\frac{(1+\kappa_1)[2A(\kappa_1-1)\xi-(3A\kappa_1+B)x_2]+(3-\kappa_1)[x\{5A(\kappa_1-2)+B\}}{(1+\kappa_1)r_2^4}$$

$$\frac{+\xi\{A(7\kappa_1+4)+B\}]}{(1+\kappa_1)r_2^4}+16A\frac{(1+\kappa_1)[x_2\{\xi^2+(\kappa_1+7)x\xi+\kappa_1x^2\}]}{(1+\kappa_1)r_2^6}$$

$$\frac{-(3-\kappa_1)[x_2\xi\{6\xi+(\kappa_1-5)x_2\}+y_1^2\{\kappa_1x_2-2x\}]}{(1+\kappa_1)r_2^6}$$

$$-192Ax_2x\xi\frac{(1+\kappa_1)x_2^2+(3-\kappa_1)y_1^2}{(1+\kappa_1)r_2^8}\Bigg], \tag{15}$$

(B) In the case that a point (x, y) exists in material 2

$$\sigma_{x2}^D=\frac{DG_1}{2\pi(\kappa_1-1)}\Bigg[2\frac{A(\kappa_1^2-2\kappa_1+3)+2(\kappa_1-B)-\kappa_1^2-1}{(1+\kappa_1)r_1^2}$$

$$+2\frac{(1+\kappa_1)[\{\kappa_1(1-A)-(1-B)\}y_1^2-2x_1\{\xi(1-A)+x(B-1)\}]}{(1+\kappa_1)r_1^4}$$

$$\frac{-(3-\kappa_1)x_1[2(B-A)\xi+x_1\{(1-A)\kappa_1-(1-B)\}}{(1+\kappa_1)r_1^4}$$

$$+\frac{2\{(x-3\xi)(A-B)-3(1-B)x_1\}]}{(1+\kappa_1)r_1^4}$$

$$+16x_1\frac{(1+\kappa_1)\{\xi(1-A)+x(B-1)\}y_1^2+(3-\kappa_1)\{\xi(1-A)+x(B-1)\}x_1^2}{(1+\kappa_1)r_1^6}\Bigg], \tag{16}$$

$$\sigma_{y2}^D=\frac{DG_1}{2\pi(\kappa_1-1)}\Bigg[2\frac{\kappa_1^2(1-A)+3(A-2B+1)}{(1+\kappa_1)r_1^2}$$

$$+2\frac{(1+\kappa_1)[2x_1\{\xi(1-A)+x(B-1)\}+y_1^2\{\kappa_1(A-1)+3(B-1)\}]}{(1+\kappa_1)r_1^4}$$

$$+\frac{(3-\kappa_1)[x_1[2(A-B)\xi+x_1\{(1-A)\kappa_1+B-1\}]+2y_1^2(1-A)]}{(1+\kappa_1)r_1^4}$$

$$-32x_1y_1^2\frac{\{\xi(1-A)+x(B-1)\}(\kappa_1-1)}{(1+\kappa_1)r_1^6}\Bigg], \tag{17}$$

$$\tau_{xy2}^D=\frac{DG_1y_1}{2\pi(\kappa_1-1)}\Bigg[4\frac{(B-A)\{\xi(3\kappa_1-5)+x(3-\kappa_1)\}-2x_1\{(1-A)\kappa_1+3(1-B)\}}{(1+\kappa_1)r_1^4}$$

$$+16x_1\frac{(1-B)\{x_1^2(1+\kappa_1)+y_1^2(3-\kappa_1)\}+2(1-\kappa_1)(B-A)x_1\xi}{(1+\kappa_1)r_1^6}\Bigg], \tag{18}$$

where $r_1^2, r_2^2, x_1, x_2, y_1$ are defined as follows

$$r_1^2=x_1^2+y_1^2,\ r_2^2=x_2^2+y_1^2,\ x_1=(x-\xi),\ x_2=(x+\xi),\ y_1=(y-\eta) \tag{19}$$

and κ_i, A, B, Γ is defined as follows.

$$\kappa_i=\begin{Bmatrix} 3-4\nu_i & (\text{Plane strain } i=1,2) \\ \dfrac{3-\nu_i}{1+\nu_i} & (\text{Plane stress } i=1,2) \end{Bmatrix}, \tag{20}$$

$$A=\frac{1-\Gamma}{\Gamma\kappa_1+1},\quad B=\frac{\kappa_2-\Gamma\kappa_1}{\Gamma+\kappa_2},\quad \Gamma=\frac{G_2}{G_1}. \tag{21}$$

When a point force acts to a point (ξ,η) in material 2, the notations $\kappa_1, \kappa_2, \Gamma$ in (1)–(12), (21) must be replaced by κ_2, κ_1, $1/\Gamma$ respectively. When a displacement discontinuity acts to a point (ξ,η) in material 2, they must be replaced in a similar manner.

3. Method of analysis

3.1. Outline of the analysis method

Figure 3 shows the present analysis method. Consider the imaginary boundaries where body forces or displacement discontinuities are distributed in an infinite plate having the interface. The body forces means continuously embedded point force. The given boundary conditions are

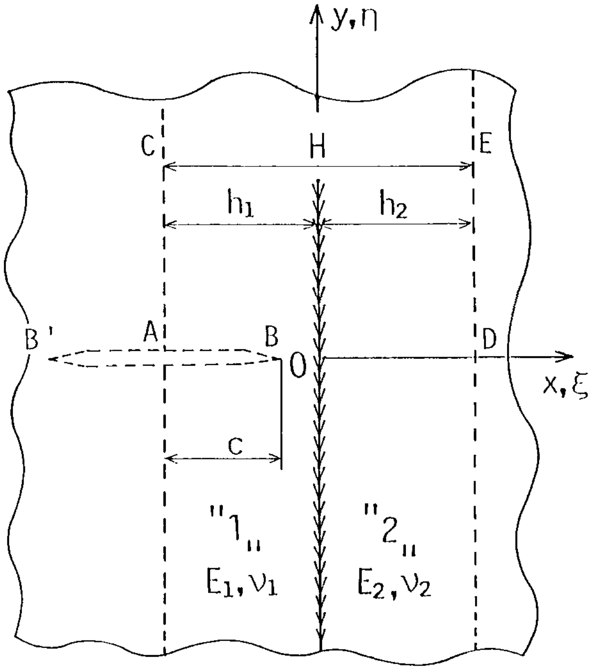


Fig. 3. Imaginary boundaries where body forces or displacement discontinuities are distributed.

satisfied by applying body force and displacement discontinuities along the prospective sites of stress free boundaries for edge and crack respectively. Therefore the method of analysis is reduced to determining the densities of body force and displacement discontinuity. In this analysis, the displacement discontinuities are symmetrically distributed not only along the boundary AB but also along the boundary AB' in order to make the shear stress which will appear at the boundary AC small in advance. The boundary length in the y-direction AC and DE are determined from the condition that the calculated results virtually do not change by increasing its length. The minimum value of the length AC is about two times the width H. In this paper, the solutions are obtained by superposing the stress fields of body forces and displacement discontinuities so as to satisfy a given boundary condition. The detail for the numerical procedure may be found in [13, 14, 17].

3.2. *Definition of stress intensity factor and the boundary conditions given at infinity*

In this paper, the stress intensity factors in two bonded elastic layers with a single edge crack under various boundary conditions at infinity are analyzed. The dimensionless stress intensity factor F_{I} is defined as follows

$$F_{\mathrm{I}} = \frac{K_{\mathrm{I}}}{\sigma_0\sqrt{\pi c}}, \tag{22}$$

where K_{I} is stress intensity factor, σ_0 is a constant corresponding to the magnitude of the stress at infinity. Now we define the stresses at infinity in material 1 and in material 2 as σ_{y1_∞} and

$\sigma_{y2\infty}$ respectively. The relation between $\sigma_{y1\infty}$ and $\sigma_{y2\infty}$ at the interface is shown in (23) from the condition that the strains at infinity must be equal.

$$\left.\begin{aligned} \frac{\sigma_{y2\infty}|_{x=0}}{\sigma_{y1\infty}|_{x=0}} &= \frac{E_2}{E_1} \quad &\text{(Plane stress)} \\ \frac{\sigma_{y2\infty}|_{x=0}}{\sigma_{y1\infty}|_{x=0}} &= \frac{E_2(1-\nu_1^2)}{E_1(1-\nu_2^2)} \quad &\text{(Plane strain)} \end{aligned}\right\}, \tag{23}$$

where $\nu_1 = \nu_2 = 0.3$.
The stresses at infinity $\sigma_{y1\infty}$, $\sigma_{y2\infty}$ are given as follows:

(A) In the case that a single edge crack exists in material 1 ($c < h_1$)

(1) Uniform tension (Fig. 1(a))

$$\left.\begin{aligned} &\sigma_{y1\infty} = \sigma_0,\ \sigma_{y2\infty} = (E_2/E_1)\sigma_0, \\ &\sigma_0 = P/\{h_1 + (E_2/E_1)h_2\} \end{aligned}\right\}, \tag{24}$$

where P is the magnitude of external tensile force.

(2) In-plane bending (Fig. 1(b))

$$\begin{aligned} \sigma_{y1\infty} &= -\frac{\sigma_0}{h_1+e}(x-e), \\ \sigma_{y2\infty} &= -\frac{E_2}{E_1}\frac{\sigma_0}{h_1+e}(x-e), \\ \sigma_0 &= M(h_1+e)/I_g \quad (I_g \text{ is given by (38)}). \end{aligned} \tag{25}$$

The stress $\sigma_{y1\infty}$ is equal to σ_0 when $x = -h_1$. Here M is the magnitude of external bending moment at infinity and e is the distance from the interface to the neutral axis shown in Fig. 4(a) and defined as follows.

$$e = \{(E_2/E_1)h_2^2 - h_1^2\}/[2\{(E_2/E_1)h_2 + h_1\}]. \tag{26}$$

(3) Pure tension (tension at the mid-point of the ligament) (Fig. 1(c))

$$\sigma_0 = P/(h_1 + h_2 - c), \tag{27}$$

where P is the magnitude of external tensile force.

(4) Tension at the mid-point of the width H (Fig. 1(d))

$$\begin{aligned} \sigma_0 &= P/[h_1 + (E_2/E_1)h_2] + M(h_1+e)/I_g \\ &= P\left\{\frac{2I_g + [h_1 + (E_2/E_1)h_2](h_1+e)(2e - h_2 + h_1)}{2I_g[h_1 + (E_2/E_1)h_2]}\right\}, \end{aligned} \tag{28}$$

where P is the magnitude of external tensile force and the bending moment M is shown in (29).

$$M = P\{e + (h_1 - h_2)/2\}. \tag{29}$$

(B) In the case that a single edge crack crossing the interface ($c > h_1$)

(1) Uniform tension

$$\left.\begin{aligned} \sigma_{y1\infty} &= (E_1/E_2)\sigma_0, \ \sigma_{y2\infty} = \sigma_0, \\ \sigma_0 &= P/\{(E_1/E_2)h_1 + h_2\} \end{aligned}\right\}, \tag{30}$$

where P is the magnitude of external tensile force.

(2) In-plane bending

$$\begin{aligned} \sigma_{y1\infty} &= -\frac{E_1}{E_2}\frac{\sigma_0}{h_1 + e}(x - e) \\ \sigma_{y2\infty} &= -\frac{\sigma_0}{h_1 + e}(x - e), \\ &\sigma_0 = M(h_1 + e)/I_g \quad (I_g \text{ is given by (38)}). \end{aligned} \tag{31}$$

The stress $\sigma_{y1\infty}$ become $(E_1/E_2)\sigma_0$ at $x = -h_1$.

(3) Pure tension (tension at the mid-point of the ligament)

$$\sigma_0 = P/(h_1 + h_2 - c), \tag{32}$$

where P is the magnitude of external tensile force.

(4) Tension at the mid-point of the width H

$$\begin{aligned} \sigma_0 &= P/[(E_1/E_2)h_1 + h_2] + M(h_1 + e)/I_g, \\ &= P\left\{\frac{2I_g + [(E_1/E_2)h_1 + h_2](h_1 + e)(2e - h_2 + h_1)}{2I_g[(E_1/E_2)h_1 + h_2]}\right\}, \end{aligned} \tag{33}$$

where P is the magnitude of external tensile force and the bending moment M is shown in (34).

$$M = P\{e + (h_1 - h_2)/2\}. \tag{34}$$

3.3. Solution for two bonded elastic layers with a single edge crack

In the actual numerical analysis the densities of the body force and the displacement discontinuity that satisfy the boundary conditions completely for the free edges and the crack can not be obtained in the closed form. Therefore, the imaginary boundaries are divided and the problem is solved numerically. The densities of the body forces and the displacement discontinuity, which are assumed to be constant in each interval, are determined from the boundary condition at the mid-point of each interval. Thus, the values of the resultant and the bending moment at the minimum section would be different from the values of them at infinity because the stress that should be zero at the boundaries would remain slightly. In the analysis for this problem we have to estimate exactly both the resultant and the bending moment at the minimum section to get an accurate solution. Therefore, the following analysis method is used in this paper. First, a uniform tensile stress is applied as the boundary condition at infinity. Under this condition, the resultant P_1, the bending moment M_1 at the minimum section and the density of displacement discontinuities γ at the crack tip are calculated numerically. Next, a bending stesss is applied as a boundary condition at infinity. Under this condition, the resultant P_2, the bending moment M_2 and the density of displacement discontinuities γ_B are also calculated. Using both of those results, we obtain the dimensionless stress intensity factor F_{Ic} for the case in which only resultant is produced at the minimum section and the bending moment equals to zero (Fig. 4c).

(A) Pure tension

$$F_{\mathrm{Ic}} = \frac{\gamma - \alpha_1 \gamma_B}{(P_1 - \alpha_1 P_2)/A_n}, \tag{35}$$

where $\alpha_1 = M_1/M_2, A_n = h_1 + h_2 - c.$ (36)

And we can estimate the dimensionless stress intensity factor for the case in which only the bending moment is produced at the minimum section and the resultant equals to zero (Fig. 4b).

(B) In-plane bending

$$F_{\mathrm{Ib}} = \frac{\gamma_B - \alpha_2 \gamma}{(M_2 - \alpha_2 M_1)/Z_g}, \tag{37}$$

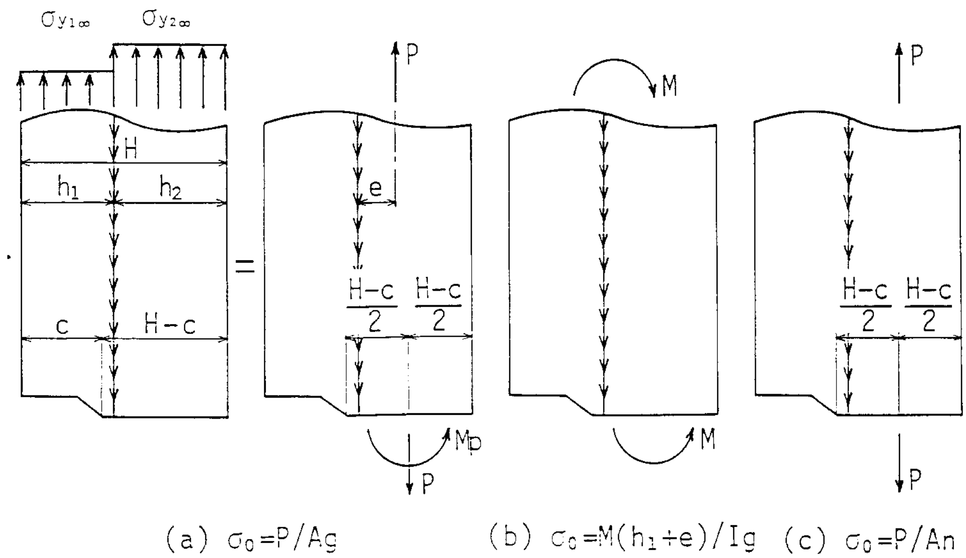


Fig. 4. Resultant and bending moment at the minimum sections under various loading conditions.

where $\alpha_2 = P_2/P_1$, $Z_g = I_g/\beta_1$, $\beta_1 = h_1 + e$,

$$I_g = \left\{ \begin{array}{l} \dfrac{\left(h_1^3 + \dfrac{E_2}{E_1}h_2^3\right)}{3} + e\left(h_1^2 - \dfrac{E_2}{E_1}h_2^2\right) + e^2\left(h_1 + \dfrac{E_2}{E_1}h_2\right) \quad (c < h_1) \\ \dfrac{\left(h_2^3 + \dfrac{E_1}{E_2}h_1^3\right)}{3} + e\left(\dfrac{E_1}{E_2}h_1^2 - h_2^2\right) + e^2\left(h_2 + \dfrac{E_1}{E_2}h_1\right) \quad (c > h_1) \end{array} \right\}. \tag{38}$$

Using (35) and (37), we can obtain the dimensionless stress intensity factor F_{Ia} for the case of uniform tension (Fig. 4a) by superposing the two loading conditions: (A) only the resultant P is produced at the minimum section (Fig. 4c) and (B) only the bending moment Mp is produced at the minimum section (Fig. 4b).

(C) Uniform tension

$$F_{Ia} = \alpha_3 F_{Ic} + \frac{M_p}{Z_g} F_{Ib}, \tag{39}$$

$$\left.\begin{array}{l} \text{where } \alpha_3 = A_g/A_n \\ A_g = \begin{cases} h_1 + (E_2/E_1)h_2 & (c < h_1) \\ h_2 + (E_1/E_2)h_1 & (c > h_1) \end{cases} \\ M_p = \begin{cases} \sigma_0\{h_1(h_2 + c) + E_2/E_1(c - h_1)h_2\}/2 & (c < h_1) \\ \sigma_0\{E_1/E_2(h_2 + c)h_1 + (c - h_1)h_2\}/2 & (c > h_1) \end{cases} \end{array}\right\}. \tag{40}$$

Moreover, we can estimate the dimensionless stress intensity factor F_{Id} for the case in which the external load P is applied at mid-point of the plate at infinity.

(D) Tension at the mid-point of the width H

$$F_{Id} = \left\{ \begin{array}{ll} \dfrac{\alpha_4 F_{Ib} + F_{Ia}}{\alpha_4 + 1} & (c < h_1) \\ \alpha_4 F_{Ib} + F_{Ia} & (c > h_1) \end{array} \right\}, \tag{41}$$

where

$$\alpha_4 = \left\{ \begin{array}{l} \left\{e + \dfrac{(h_1 - h_2)}{2}\right\} \dfrac{h_1 + (E_2/E_1)h_2}{I_g}(h_1 + e) \quad (c < h_1) \\ \left\{e + \dfrac{(h_1 - h_2)}{2}\right\} \dfrac{(E_1/E_2)h_1 + h_2}{I_g}(h_1 + e) \quad (c > h_1) \end{array} \right\}. \tag{42}$$

4. Results and discussion

4.1. Stress intensity factor of a homogeneous strip with a single edge crack

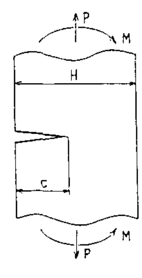
If we put the moduli of direct elasticity of both materials equal to: $E_1 = E_2(\nu_1 = \nu_2)$. Hetenyi's solution used as the fundamental solutions in this analysis becomes the solutions for homogeneous infinite plate, namely, Kelvin's solution. Then, we can obtain the stress intensity factor of a homogeneous strip with a single edge crack. In Table 1, the present results (BFM) are compared with the values given by Kaya and Erdogan [15]. The values of BFM in Table 1 are determined by the extrapolation method [16, 17]. The two results coincide with each other in the 4 significant digits and the relative error is within 0.15 percent.

4.2. Stress intensity factor of a surface layer bonded to a semi-infinite plane with an edge crack

Table 2 shows the stress intensity factors of a surface layer bonded to a semi-infinite plane with an edge crack. The results in Table 2 are plotted in Fig. 5. In Table 2, the values in the parentheses are analyzed for plane stress condition. The other values in Table 2 are analyzed for plane strain condition. As the crack length increases, the difference of both results for plane

Table 1. SIFs of an edge crack in a homogeneous strip ($E_1 = E_2$)

	Uniform tension		In-plane bending	
c/H	HIEM	BFM ($E_1 = E_2$)	HIEM	BFM ($E_1 = E_2$)
0.1	1.1892	1.189	1.0472	1.046
0.2	1.3673	1.367	1.0553	1.054
0.3	1.6599	1.659	1.1241	1.123
0.4	2.1114	2.111	1.2606	1.259
0.5	2.8246	2.823	1.4972	1.495
0.6	4.0332	4.032	1.9140	1.913
0.7	6.3549	6.355	2.7252	2.725
0.8	11.955	11.95	4.6764	4.675
0.9	34.633	34.62	12.462	12.46

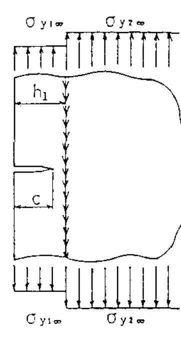


$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = P/H$ (Tension)
$\sigma_0 = 6M/H^2$ (Bending)

Table 2. SIFs of an edge crack in a surface layer bonded to a semi-infinite plane

c/h_1	$E_2/E_1 = 1/3$	$E_2/E_1 = 3.0$	$E_2/E_1 = \infty$
0.1	1.133	1.113 (1.113)	1.106
0.2	1.160	1.095 (1.095)	1.074
0.3	1.195	1.073 (1.073)	1.034
0.4	1.234	1.049 (1.048)	0.992
0.5	1.276	1.024 (1.021)	0.946
0.6	1.320	0.996 (0.991)	0.896
0.7	1.371	0.964 (0.956)	0.839
0.8	1.437	0.922 (0.911)	0.767
0.9	1.552	0.857 (0.841)	0.658
0.925	1.604	0.832	0.617
0.95	1.682	0.797 (0.776)	0.563
0.975	1.831	0.741	0.482
0.995	2.280	0.623 (0.594)	0.337

[Plane strain, ( ): Plane stress]



$F_I = K_I/\sigma_{y1\infty}\sqrt{\pi c}$
$\sigma_{y2\infty} = E_2/E_1 \cdot \sigma_{y1\infty}$
$\nu_1 = \nu_2 = 0.3$

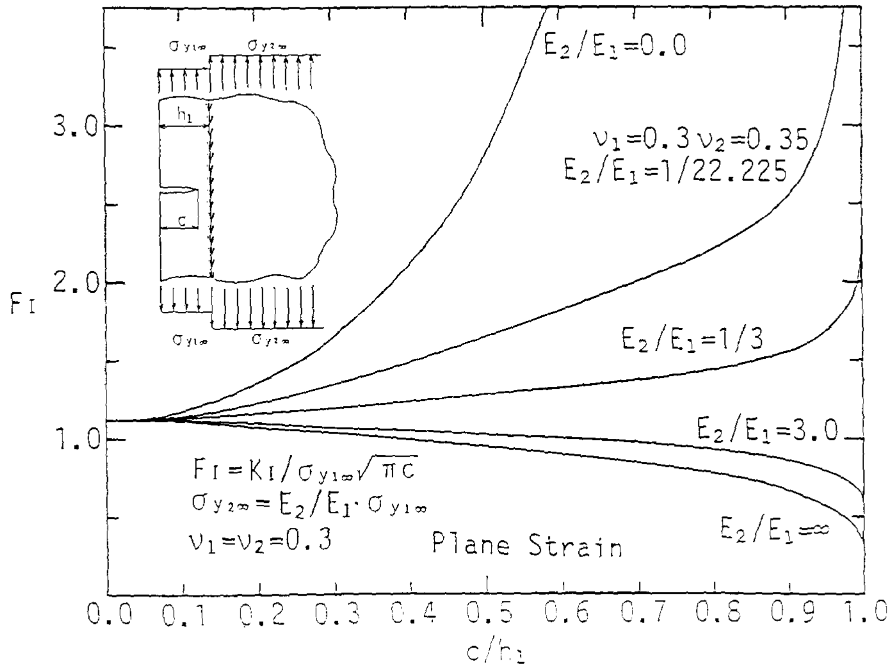


Fig. 5. SIFs of an edge crack in a surface layer bonded to a semi-infinite plane.

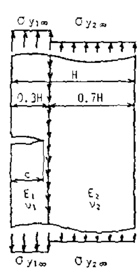
stress and plane strain increases. However, even in the case that the relative crack length $c/h_1 = 0.995$, the relative error is only about 5 percent. In this paper, the stress intensity factors for plane strain condition are mainly shown. The values from the chart given by Gecit [18] are in good agreement with the present results shown in Table 2 within 2 percent.

4.3. *Stress intensity factor of two bonded elastic layers with a single edge crack* ($c < h_1$)

In Table 3, the values given by [10] using the boundary element method (BEM) are compared with the present results when the crack tip exists near the interface in the two bonded elastic layers. As the crack tip approaches the interface exceedingly fast, the solution of the BEM has the relative error of about 6 percent. Table 4 shows the stress intensity factors of two bounded elastic layers with a single edge crack under uniform tension. The results in Table 4 are plotted in Figs. 6 and 7. The values for $E_2/E_1 = \infty$ in Table 4 coincide with the values for $E_2/E_1 = \infty$

Table 3. Comparison of the results of analysis given by BFM and BEM ($c < h_1$)

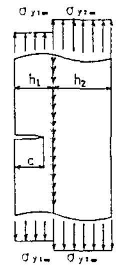
c/H	BFM	BEM
0.2	1.572	1.549
0.25	1.868	1.855
0.28	2.191	2.238
0.285	2.281	2.340
0.29	2.402	2.504
0.293	2.506	2.637
0.295	2.605	2.754
0.297	2.753	2.919



$F_I = K_I/\sigma_{y1\infty}\sqrt{\pi c}$
$\sigma_{y1\infty} = E_1/E_2 \cdot \sigma_{y2\infty}$
$E_2/E_1 = 1/2$
$\nu_1 = \nu_2 = 0.3$
Plane stress

Table 4. SIFs of two bonded elastic layers with a single edge crack under uniform tension ($c < h_1$)

	$E_2/E_1 = 3.0$		$E_2/E_1 = 1/3$		$E_2/E_1 = \infty$	
c/h_1 \ h_2/h_1	1.0	3.0	1.0	3.0	1.0	3.0
0.05	1.122	1.118	1.128	1.125		
0.1	1.129	1.116	1.152	1.138		
0.2	1.154	1.107	1.232	1.181	1.074	1.074
0.4	1.250	1.090	1.511	1.319	0.992	0.992
0.6	1.390	1.071	1.984	1.527	0.896	0.896
0.8	1.528	1.031	2.817	1.859	0.767	0.767
0.9	1.538	0.976	3.589	2.163	0.658	0.658
0.95	1.481	0.916	4.300	2.456		
0.995	1.198	0.792	6.566	3.581	0.337	0.337



$F_I = K_I/\sigma_{y1\infty}\sqrt{\pi c}$
$\sigma_{y2\infty} = E_2/E_1 \cdot \sigma_{y1\infty}$
$\nu_1 = \nu_2 = 0.3$
Plane strain

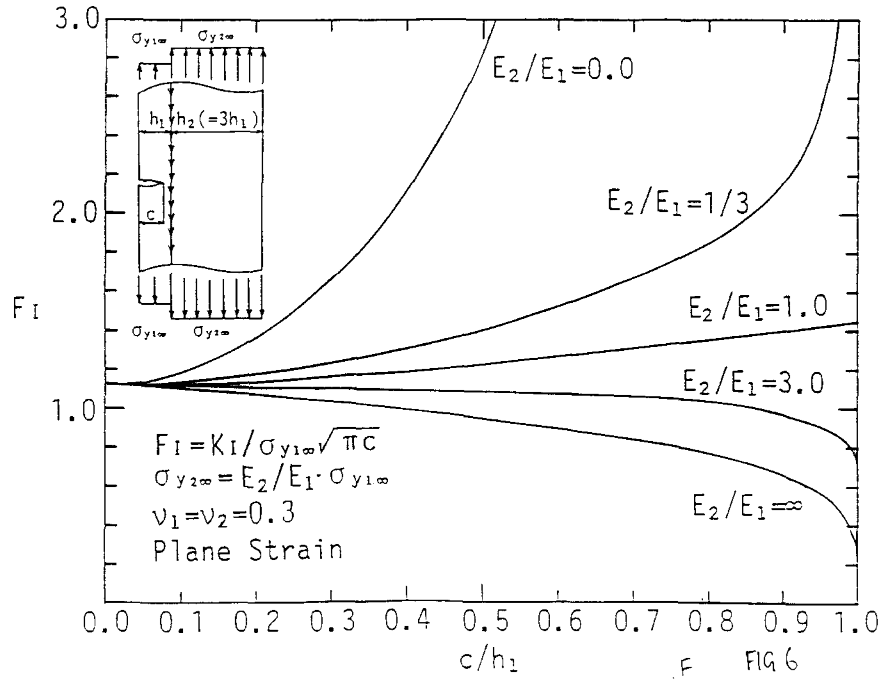


Fig. 6. SIFs of two bonded elastic layers with a single edge crack under uniform tension ($h_2 = 3h_1$).

in Table 2. In Tables 5–7, the stress intensity factors of the same composite material in Table 4 under in-plane bending, pure tension and tension at mid-point of width H at infinity are shown. The results in Tables 5–7 are plotted in Figs. 8–13.

The stress intensity factors which are normalized by the surface stress at the crack side σ_0 and the crack length c are shown in Figs. 6–9, and 12, 13. The normalized stress intensity factors go to infinity when $E_1 > E_2$ and they decrease to zero when $E_1 < E_2$. Moreover, when the crack length decreases, the stress intensity factors approach the value 1.1215 for the homogeneous semi-infinite plane having an edge crack.

4.4. Stress intensity factor of two bonded elastic layers with a single edge crack crossing the interface ($c > h_1$)

In Table 8, the values given by [10] using the BEM are compared with the present results when the crack tip crosses the interface in the two bonded elastic layers. Both of them are different by

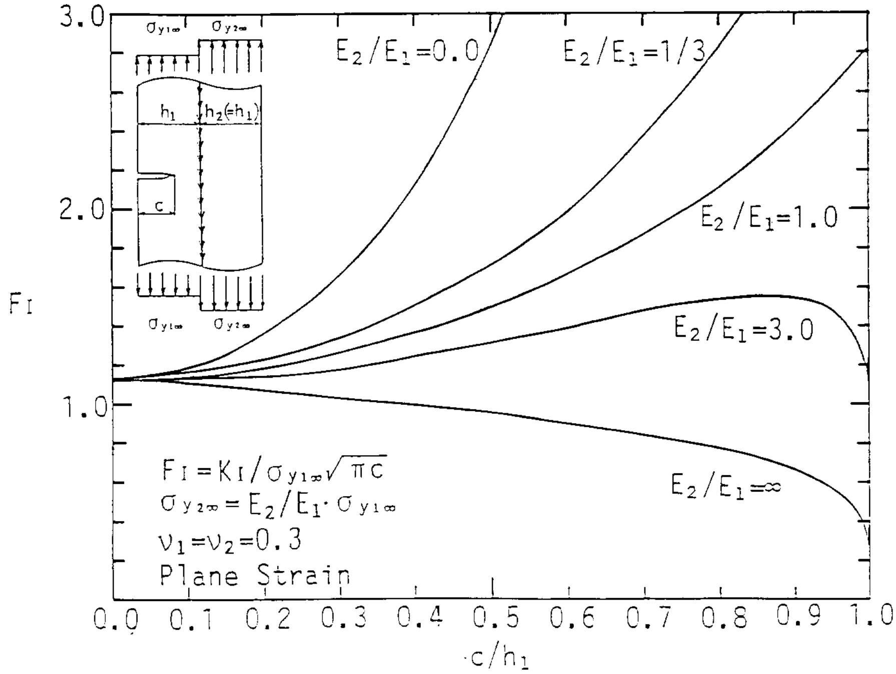
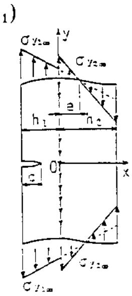


Fig. 7. SIFs of two bonded elastic layers with a single edge crack under uniform tension ($h_2 = h_1$).

about 16 percent for the worst case. However, the results in the present analysis approach the solutions of the deep crack smoothly as the relative crack length $c/H \rightarrow 1$. Tables 9–12 show the stress intensity factors for the fixed crack length $c = 1.2h_1$ under uniform tension, in-plane bending, pure tension and tension at the mid-point of the width H. The results in Tables 9–12 are plotted in Figs. 14–17 respectively. When the crack tip crosses the interface and the relative crack length c/H is large enough, the stress intensity factor can be estimated with good accuracy by using the solution of the deep crack. The solutions of the deep crack [21]

Table 5. SIFs of two bonded elastic layers with a single edge crack under in-plane bending ($c < h_1$)

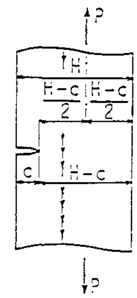
	$E_2/E_1 = 3.0$		$E_2/E_1 = 1/3$		$E_2/E_1 = \infty$	
c/h_1 \ h_2/h_1	1.0	3.0	1.0	3.0	1.0	3.0
0.05	1.094	1.103	1.082	1.102		
0.1	1.074	1.086	1.059	1.092		
0.2	1.043	1.048	1.038	1.087	0.985	1.020
0.4	1.015	0.973	1.065	1.116	0.823	0.890
0.6	1.012	0.898	1.171	1.189	0.659	0.754
0.8	0.995	0.808	1.393	1.336		
0.9	0.946	0.738	1.619	1.494	0.378	0.490
0.95	0.886	0.680	1.846	1.660		
0.995	0.701	0.535	2.680	2.369	0.177	0.241



$F_1 = K_1/\sigma_0\sqrt{\pi c}$
$\sigma_0 = \sigma_{y1\infty}|_{x=-h_1}$
$\nu_1 = \nu_2 = 0.3$
Plane strain

Table 6. SIFs of two bonded elastic layers with a single edge crack under pure tension ($c < h_1$)

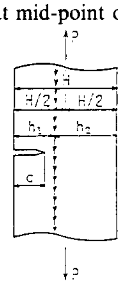
c/h_1 \ h_2/h_1	$E_2/E_1 = 3.0$ 1.0	3.0	$E_2/E_1 = 1/3$ 1.0	3.0	$E_2/E_1 = 1.0$ 1.0	3.0
→ 0.0	1.208	0.745	0.518	1.056	0.793	0.561
0.05	1.101	0.706	0.445	1.012		
0.1	1.007	0.669	0.388	0.980	0.657	0.509
0.2	0.844	0.596	0.305	0.933	0.557	0.465
0.4	0.594	0.469	0.221	0.886	0.415	0.394
0.6	0.405	0.364	0.211	0.882	0.321	0.338
0.8	0.252	0.273	0.279	0.937	0.254	0.294
0.9	0.183	0.227	0.372	1.025		
0.95	0.147	0.199	0.467	1.132		
0.995	0.111	0.149	0.768	1.621		
→ 1.0					0.205	0.257



$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = P/(H - c)$
$\nu_1 = \nu_2 = 0.3$
Plane strain

Table 7. SIFs of two bonded elastic layers with a single edge crack under tension at mid-point of width $H(c < h_1)$

c/h_1 \ h_2/h_1	$E_2/E_1 3.0$ 1.0	3.0	$E_2/E_1 = 1/3$ 1.0	3.0
0.05	1.107	1.112	1.232	1.150
0.1	1.100	1.104	1.361	1.190
0.2	1.095	1.084	1.669	1.288
0.4	1.124	1.043	2.515	1.548
0.6	1.187	1.002	3.814	1.906
0.8	1.242	0.942	6.021	2.447
0.9	1.221	0.881	8.021	2.917
0.95	1.163	0.822	9.820	3.352
0.995	0.931	0.690	15.31	4.944



$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = P/[h_1 + (E_2/E_1)h_2] + M(h_1 + e)/I_g$
$\nu_1 = \nu_2 = 0.3$
Plane strain

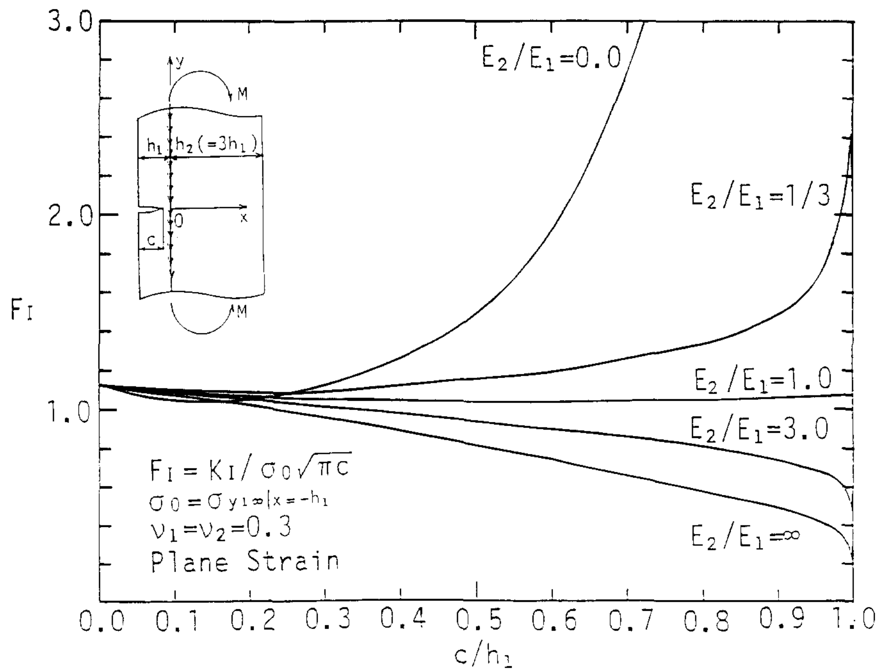


Fig. 8. SIFs of two bonded elastic layers with a single edge crack under in-plane bending ($h_2 = 3h_1$).

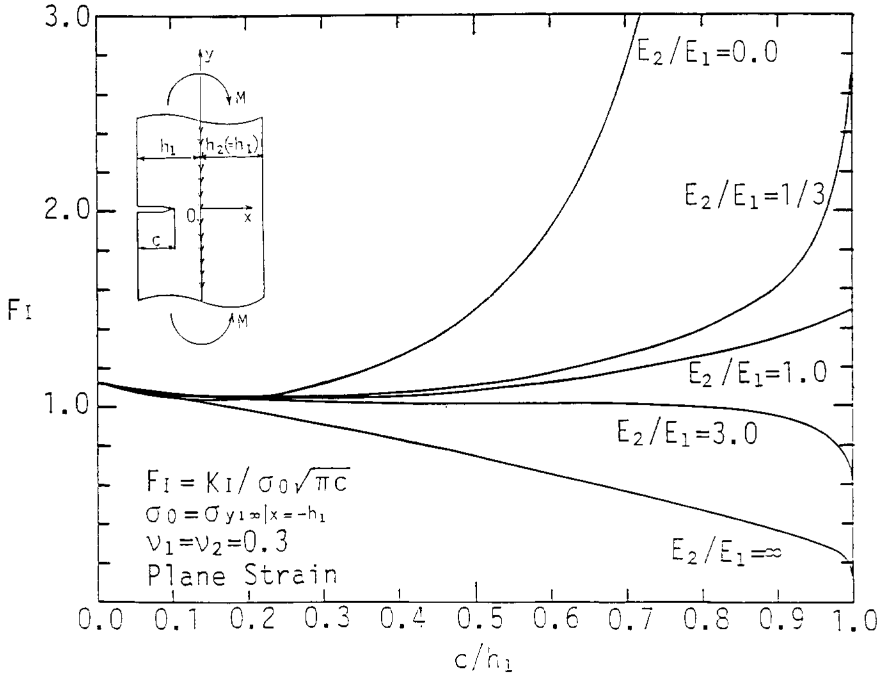


Fig. 9. SIFs of two bonded elastic layers with a single edge crack under in-plane bending ($h_2 = h_1$).

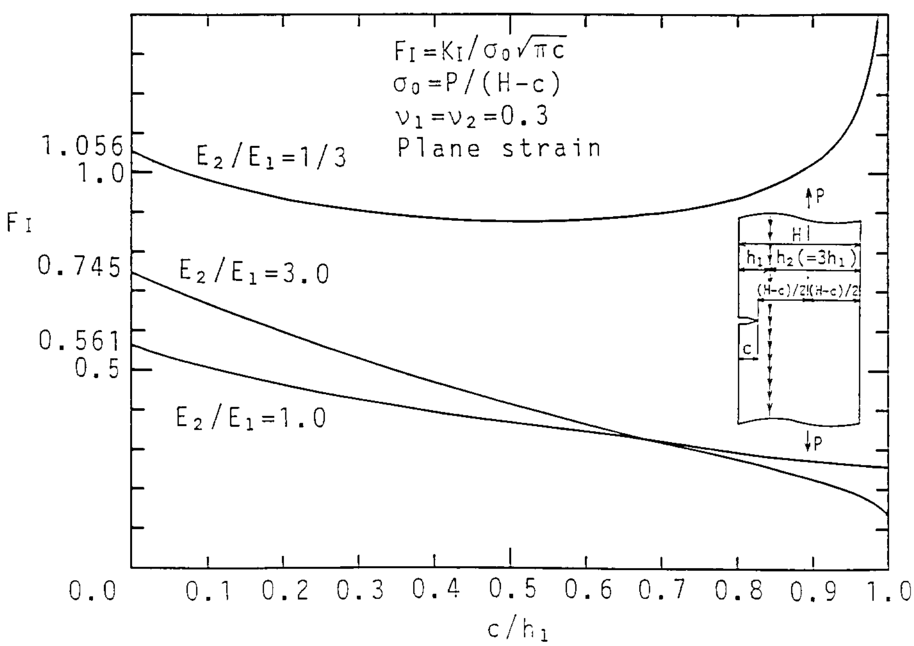


Fig. 10. SIFs of two bonded elastic layers with a single edge crack under pure tension ($h_2 = 3h_1$).

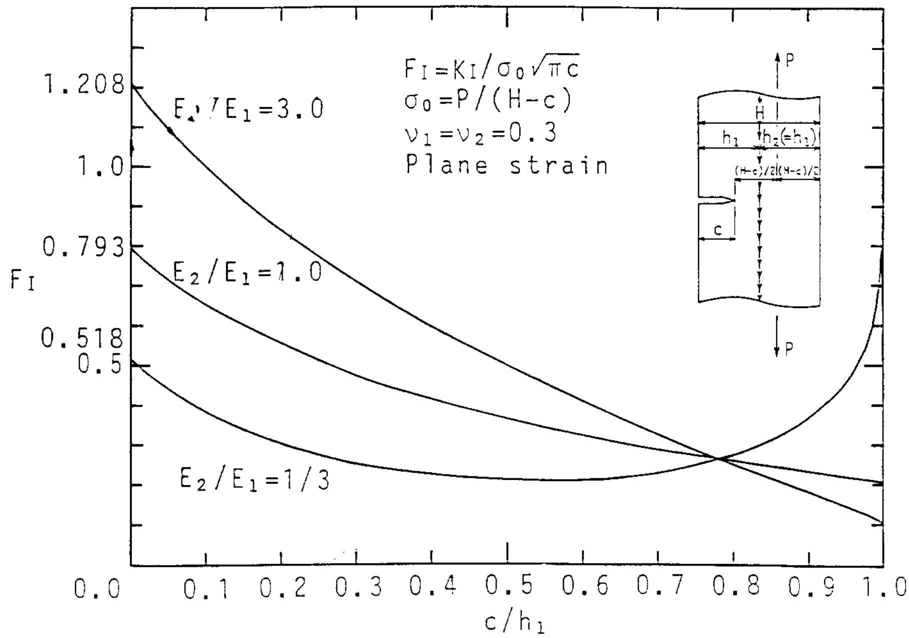


Fig. 11. SIFs of two bonded elastic layers with a single edge crack under pure tension ($h_2 = h_1$).

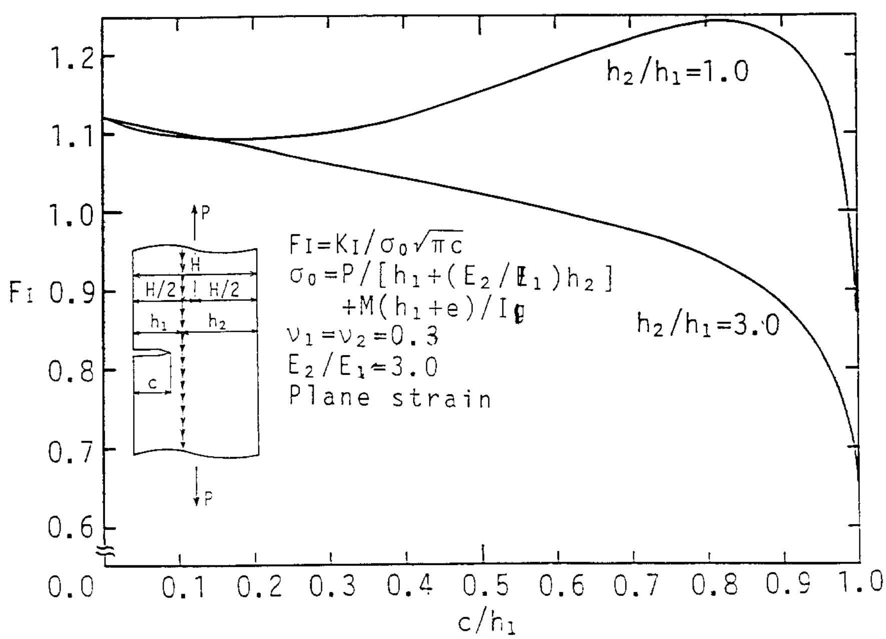


Fig. 12. SIFs of two bonded elastic layers with a single edge crack under tension at mid-point of width $H(E_2 = 3E_1)$.

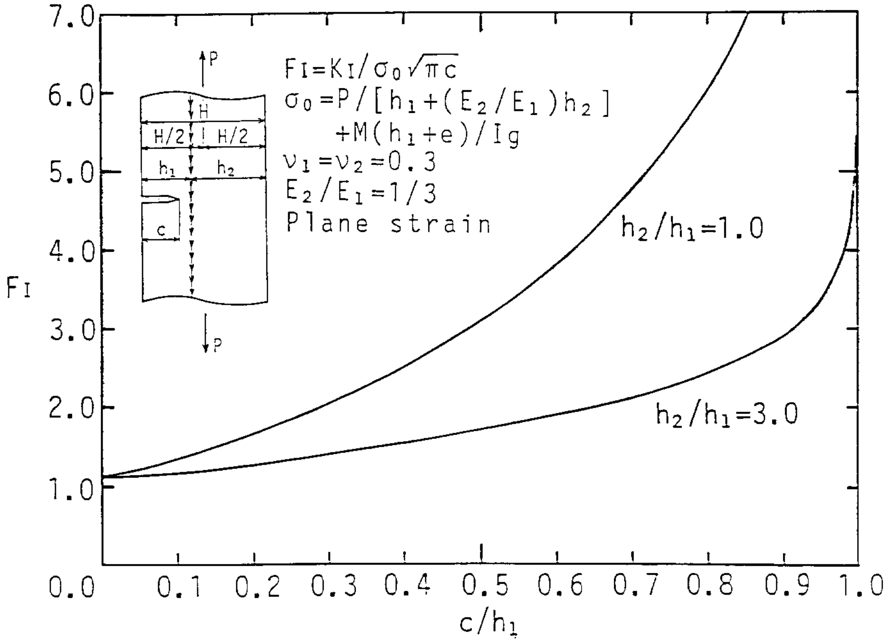
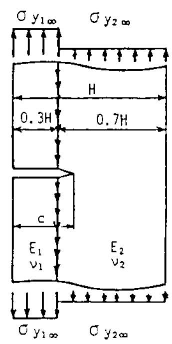


Fig. 13. SIFs of two bonded elastic layers with a single edge crack under tension at mid-point of width $H (E_2 = E_1/3)$.

Table 8. Comparison of the results of analysis given by BFM, BEM and the solution of deep crack ($c > h_1$)

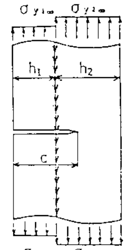
c/H	BFM	BEM	Deep crack
0.303	3.561	4.093	2.952
0.305	3.451	3.992	2.961
0.307	3.386	3.927	2.971
0.31	3.330	3.821	2.986
0.315	3.282	3.785	3.012
0.32	3.261	3.745	3.038
0.35	3.314	3.801	3.212
0.4	3.613	4.113	3.571
0.6	6.453		6.451
0.8	18.45		18.49
0.9	52.76		52.87



$F_I = K_I/\sigma_{y2\infty}\sqrt{\pi c}$
$\sigma_{y2\infty} = E_2/E_1 \cdot \sigma_{y1\infty}$
$E_2/E_1 = 1/2$
$\nu_1 = \nu_2 = 0.3$
Plane stress

Table 9. SIFs of two bonded elastic layers with a single edge crack crossing the interface under uniform tension ($c = 1.2h_1$)

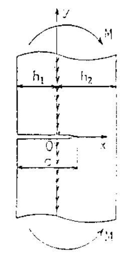
		BFM		Deep crack	
E_2/E_1					
h_2/h_1	c/H	1/3	3.0	1/3	3.0
0.5	0.8	34.01	4.526	34.13	4.565
1.0	0.6	11.10	1.701	10.94	1.738
2.0	0.4	5.444	1.019	5.246	1.092
3.0	0.3	4.072	0.868	3.853	0.975
4.0	0.24	3.466	0.807	3.237	0.951



$F_I = K_I/\sigma_{y2\infty}\sqrt{\pi c}$
$\sigma_{y1\infty} = E_1/E_2 \cdot \sigma_{y2\infty}$
$\nu_1 = \nu_2 = 0.3$
$c = 1.2h_1$
Plane strain

Table 10. SIFs of two bonded elastic layers with a single edge crack crossing the interface under in-plane bending ($c = 1.2h_1$)

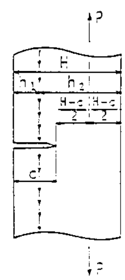
E_2/E_1		BFM		Deep crack	
h_2/h_1	c/H	1/3	3.0	1/3	3.0
0.5	0.8	10.25	1.961	10.29	1.977
1.0	0.6	4.203	0.811	4.133	0.827
2.0	0.4	2.899	0.583	2.788	0.635
3.0	0.3	2.602	0.562	2.476	0.661
4.0	0.24	2.461	0.568	2.336	0.717



$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = \sigma_{y1\infty|x=-h_1}$
$\nu_1 = \nu_2 = 0.3$
$c = 1.2h_1$
Plane strain

Table 11. SIFs of two bonded elastic layers with a single edge crack crossing the interface under pure tension ($c = 1.2h_1$)

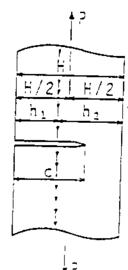
E_2/E_1		BFM			Deep crack		
h_2/h_1	c/H	1/3	3.0	1.0	1/3	3.0	1.0
0.5	0.8	0.150	0.146	0.146	0.148	0.148	0.148
1.0	0.6	0.241	0.235	0.236	0.242	0.242	0.242
2.0	0.4	0.373	0.346	0.359	0.363	0.363	0.363
3.0	0.3	0.486	0.417	0.454	0.452	0.452	0.452
4.0	0.24	0.585	0.465	0.528	0.527	0.527	0.527



$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = P/(H - c)$
$\nu_1 = \nu_2 = 0.3$
$c = 1.2h_1$
Plane strain

Table 12. SIFs of two bonded elastic layers with a single edge crack crossing the interface under tension at mid-point of width $H(c = 1.2h_1)$

E_2/E_1		BFM		Deep crack	
h_2/h_1	c/H	1/3	3.0	1/3	3.0
0.5	0.8	27.81	6.587	27.89	6.642
1.0	0.6	8.187	2.637	8.076	2.692
2.0	0.4	3.681	1.538	3.551	1.657
3.0	0.3	2.694	1.239	2.542	1.412
4.0	0.24	2.300	1.099	2.131	1.319



$F_I = K_I/\sigma_0\sqrt{\pi c}$
$\sigma_0 = P/[(E_1/E_2)h_1 + h_2] + M(h_1 + e)/I_g$
$\nu_1 = \nu_2 = 0.3$
$c = 1.2h_1$
Plane strain

are shown in (43), (44).

$$F_{ID} = \left\{ \begin{array}{ll} \alpha_5 F_{IcD} + \beta_2 F_{IbD} & \text{(Uniform tension)} \\ \alpha_6 F_{IbD} & \text{(In-plane bending)} \\ (1-\lambda) F_{IcD} & \text{(Pure tension)} \\ \alpha_5 F_{IcD} + \beta_3 F_{IbD} & \text{(Tension at mid-point of width)} \end{array} \right\}, \quad (43)$$

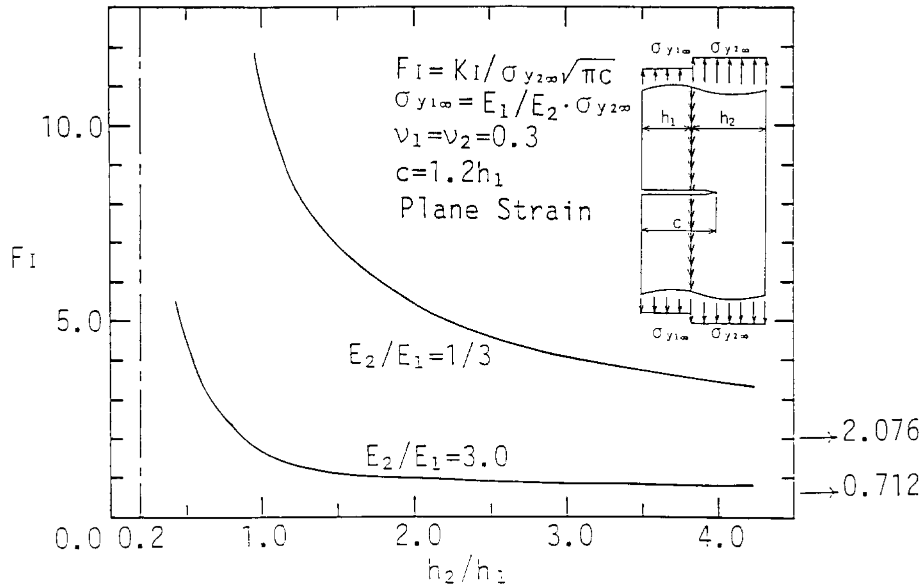


Fig. 14. SIFs of two bonded elastic layers with a single edge crack crossing the interface under uniform tension ($c = 1.2h_1$).

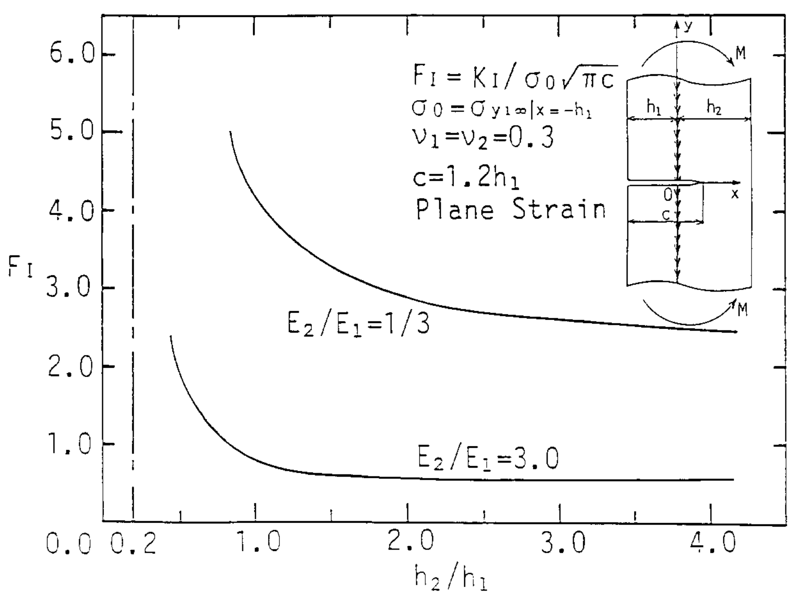


Fig. 15. SIFs of two bonded elastic layers with a single edge crack crossing the interface under in-plane bending ($c = 1.2h_1$).

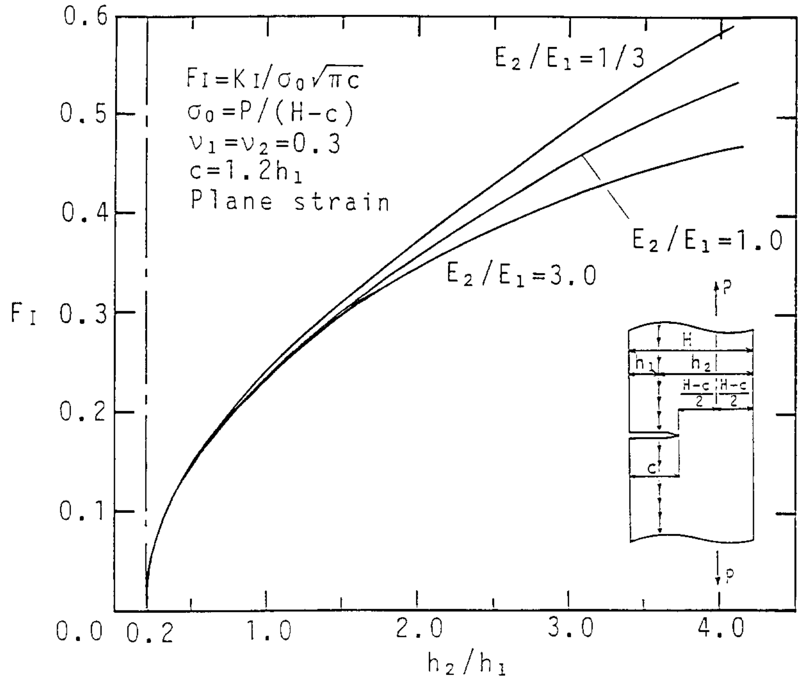


Fig. 16. SIFs of two bonded elastic layers with a single edge crack crossing the interface under pure tension ($c = 1.2h_1$).

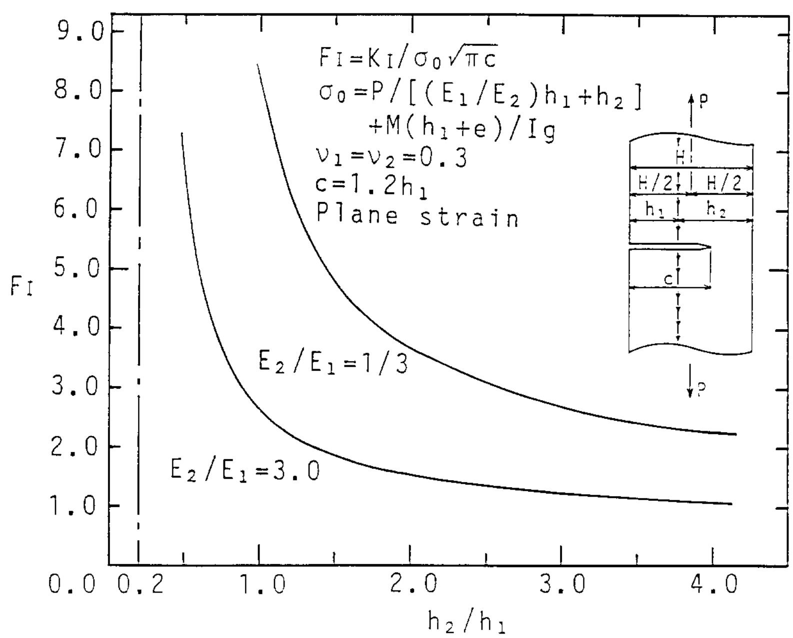


Fig. 17. SIFs of two bonded elastic layers with a single edge crack crossing the interface under tension at mid-point of width $H(c = 1.2h_1)$.

where α_5, α_6, F_{IcD}, F_{IbD}, β_2, β_3 are defined as follows.

$$\left.\begin{aligned}
&\alpha_5 = \{(E_1/E_2)h_1 + h_2\}/H, \quad \alpha_6 = 6I_g/[(h_1 + e)H^2] \\
&F_{IcD} = \frac{2}{\sqrt{\lambda}}\left\{\sqrt{\frac{1}{(\pi^2 - 4)(1 - \lambda)}} - \beta\left(\frac{2}{\beta\sqrt{\pi^2 - 4}} - 0.5\right)\sqrt{\frac{1}{(1 - \lambda)}}\right\} \\
&F_{IbD} = \beta/[3\sqrt{\lambda(1 - \lambda)^3}], \ \beta_2 = 3\alpha_5\{H - 2(h_1 + e) + c\}/H^2 \\
&\beta_3 = 3\alpha_5 c/H^2 \quad (\lambda = c/H, \ \beta = 1.1215)
\end{aligned}\right\} \tag{44}$$

5. Conclusion

In this paper, the stress intensity factors in two bonded elastic layers with a single edge crack under various loading conditions are analyzed by BFM. The stress fields induced by a point force and a displacement discontinuity in two bonded elastic half planes, which were obtained by Hetenyi's solution, are used as fundamental solutions to solve those problems. The conclusions can be made as follows:

(1) The stress intensity factors in two bonded elastic layers with a single edge crack under various loading conditions were calculated when the crack length, the stiffness ratio of materials and the plate width were changed systematically. The exact results are shown in Tables 1–12 and Figs. 5–17.
(2) The results calculated for a homogeneous strip coincide with the results by Kaya and Erdogan within 0.15 percent. The results for a single edge crack in an elastic surface layer bonded to an elastic semi-infinite plane coincide with the results by Gecit within 2 percent.
(3) The stress intensity factors of a single edge crack crossing the interface can be estimated with good accuracy by using the solutions of the deep crack.

Appendix

The error estimation of the approximate formulas for the stress intensity factors of a homogeneous strip with a single edge crack

As we described in Chap. 4.1, the results of the present analysis coincide with the results of the HIEM within 0.15 percent. Therefore, we can estimate the error of the approximate formulas which have been used in the experimental research or the design until now. The approximate formulas considered here are shown in (45)–(50). Figures 18 and 19 show the ratio of the approximate formula to the exact result under uniform tension and in-plane bending. In Fig. 18, we can conclude that Brown's approximate formula [19] underestimates the stress intensity factors by about 0.6 percent for the worst case of uniform tension and Tada's approximate formula [20] overestimates the stress intensity factors when the relative crack length $c/H < 0.1$ by about 0.6 percent for the worst case of uniform tension. In Fig. 19, Tada's approximate formula is found to give an underestimated stress intensity factor by about 2.2 percent for the case of in-plane bending. However, Brown's approximate formula is also accurate enough for in-plane bending.

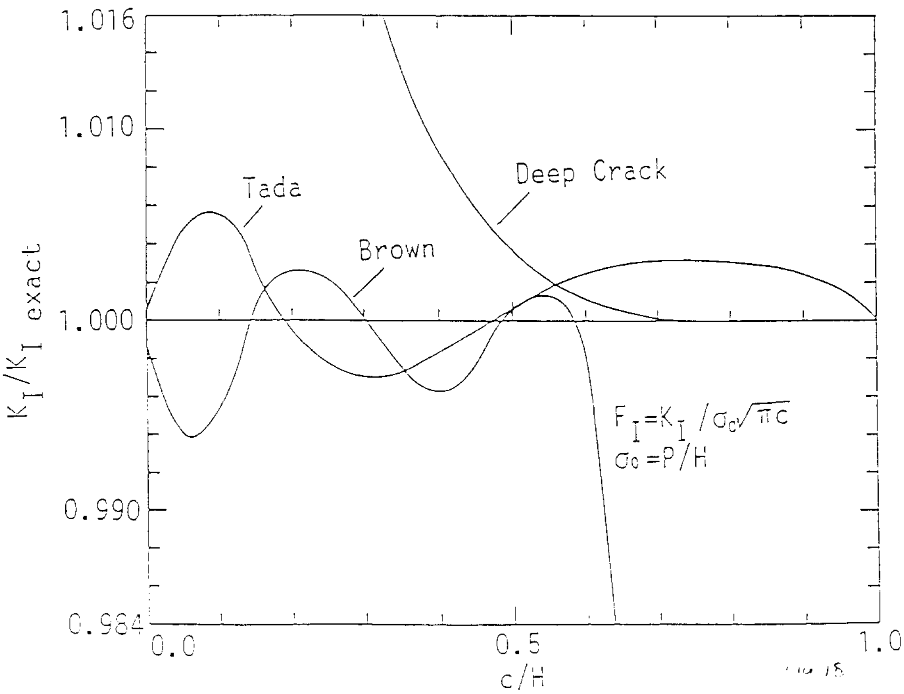


Fig. 18. Error estimation of the approximate formulas for SIFs of a homogeneous strip with a single edge crack under uniform tension.

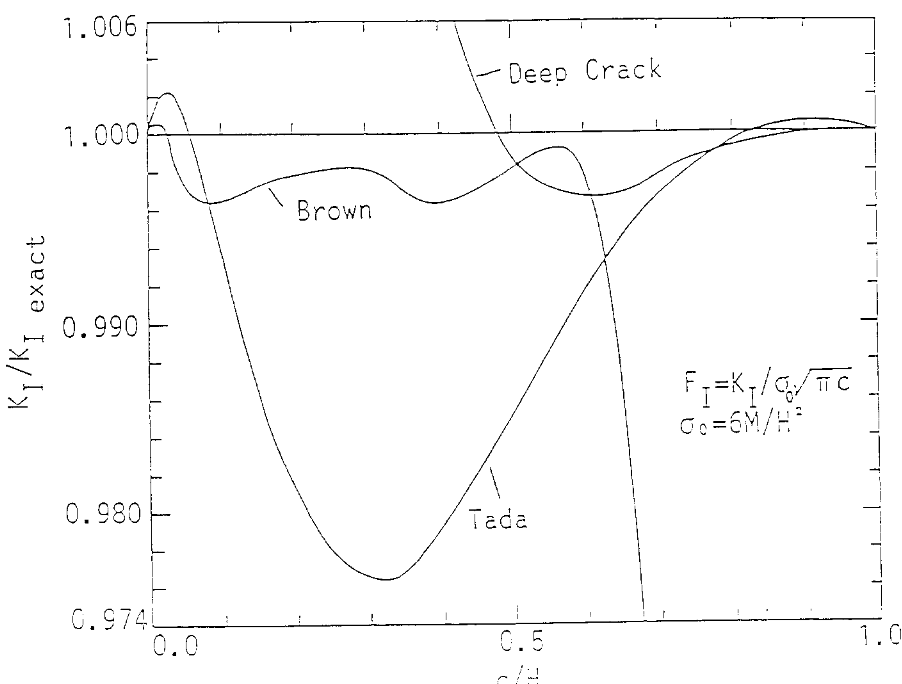


Fig. 19. Error estimation of the approximate formulas for SIFs of a homogeneous strip with a single edge crack under in-plane bending.

(A) Uniform tension

[$\sigma_0 = P/H$, P is the magnitude of external tensile force]

$$F_\mathrm{I} = 1.12 - 0.231\lambda + 10.55\lambda^2 - 21.72\lambda^3 + 30.39\lambda^4 \quad 0 \leqslant \lambda \leqslant 0.6 \quad \text{(Brown)}, \tag{45}$$

$$F_\mathrm{I} = \sqrt{\frac{2}{\pi\lambda}\tan\frac{\pi\lambda}{2}}\,\frac{0.752 + 2.02\lambda + 0.37\{1 - \sin(\pi\lambda/2)\}^3}{\cos(\pi\lambda/2)} \quad 0 \leqslant \lambda \leqslant 1.0 \quad \text{(Tada)}, \tag{46}$$

$$F_\mathrm{I} = \frac{2}{\sqrt{\lambda}}\left\{\sqrt{\frac{1}{(\pi^2 - 4)(1 - \lambda)}} - \beta\left(\frac{2}{\beta\sqrt{\pi^2 - 4}} - 0.5\right)\sqrt{\frac{1}{(1 - \lambda)}}\right\} + 3\lambda\left(\frac{\beta}{3\sqrt{\lambda(1 - \lambda)^3}}\right)$$

(Deep crack) (47)

(B) In-plane bending

[$\sigma_0 = 6M/H^2$, M is the magnitude of external bending moment]

$$F_\mathrm{I} = 1.122 - 1.40\lambda + 7.33\lambda^2 - 13.08\lambda^3 + 14.0\lambda^4 \quad 0 \leqslant \lambda \leqslant 0.6 \text{ (Brown)} \tag{48}$$

$$F_\mathrm{I} = \sqrt{\frac{2}{\pi\lambda}\tan\frac{\pi\lambda}{2}}\,\frac{0.923 + 0.199\{1 - \sin(\pi\lambda/2)\}^4}{\cos(\pi\lambda/2)} \quad 0 \leqslant \lambda \leqslant 1.0 \quad \text{(Tada)}, \tag{49}$$

$$F_\mathrm{I} = \beta/[3\sqrt{\lambda(1 - \lambda)^3}] \quad \text{(Deep crack)}, \tag{50}$$

where $\lambda = c/H$, $\beta = 1.1215$.

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