Application of Finite-Part Integrals to Planar Interfacial Fracture Problems in Three-Dimensional Bimaterials

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This paper deals with linear elastic fracture problems for a planar crack on an interface between two dissimilar elastic half-space solids bonded together. The finite-part integral concept is used to derive hypersingular integro-differential equations for the interfacial crack from the point-force solutions for a bimaterial space. Investigations on the singularities and the singular stress fields in the vicinity of the crack are made by the dominant-part analysis of the two-dimensional hypersingular integrals. Thereafter the stress intensity factor K-fields and the energy release rate G are exactly obtained by using the definitions of stress intensity factors and the principle of virtual work, respectively. The results show that, unlike the homogenous case, the asymptotic fields always consist of all three modes of fracture. Finally, some numerical examples of various aspects of elliptical cracks subjected to constant pressures are given.

Introduction

With the rapidly increasing use of composites for engineering structures, much attention has been paid to the interfacial crack by more researchers. It is useful to design and manufacture composite materials for which we know the fracture behavior of their interface. During the past few years, considerable analyses have been performed, and many problems regarding the mechanics of interfacial fracture have been discussed. However, these studies generally involved two-dimensional interfacial crack modeling; little work, theoretical work in particular, on the three-dimensional aspect of crack problems has been carried out except those of specially shaped cracks (Wills, 1972; Erdogan and Arin, 1972; Kassir and Bregman, 1972; Shibuya et al., 1989; Nakamura, 1991; Yuuki and Xu, 1992). This is mainly due to the extreme difficulties of solving such problems by mathematics and mechanics, or to the substantial computation required in the numerical analyses.

The purpose of this paper is to make more systematic and painstaking theoretical studies of a three-dimensional planar interfacial crack on a bimaterial interface by the use of hypersingular integral equation method. The key to the utilization of the hypersingular integral equation method is the availability of known point-force fundamental solutions expressed in tensor form obtained by the authors (1997) and the concept of the finite-part integral introduced by Hadamard (1952). Two-dimensional hypersingular integral equations with unknown displacement jumps across the crack surfaces are first derived for an arbitrarily shaped crack parallel to a bimaterial interface, then based on these equations, use is further made of limit theory and Taylor series expansions to obtain hypersingular integro-differential equations for the interfacial planar crack. In order to determine the stress singularities near the interfacial crack front edge, the hypersingular integro-differential equations are investigated by the dominant-part analysis of the two-dimensional hypersingular integrals. Starting strictly from the three-dimensional theory of elasticity, the dominant-part analysis of the two-dimensional hypersingular integrals is also applied to study the singular stress fields ahead of the interfacial crack front edge. The stress intensity factor K-fields for the three-dimensional planar interfacial cracks are arrived at in the light of similar definitions of K-fields for the two-dimensional interfacial cracks (Hutchinson et al., 1987; Rice, 1988). In addition, energy release rate G is analyzed by the principle of virtual work. Finally, some numerical examples of planar interfacial elliptical cracks under the normal uniform loading are given.

Hypersingular Integral Equation for a Planar Crack Parallel to a Bimaterial Interface

A fixed rectangular Cartesian coordinate system x_i is used. Subscripts of English letters always take the values 1, 2, 3 and those of Greek letters, 1, 2. The Einstein summation convention is assumed. We consider two dissimilar elastic half-spaces bonded together along the $x_1 - x_2$ plane (see Fig. 1). Suppose that the upper half-space is occupied by an elastic medium with constants (μ_1, v_1) and the lower half-space by an elastic medium with constants (μ_2, v_2), where μ is shear modulus and v Poisson's ratio. The planar crack is assumed to be located at a distance h above, and parallel to, the bimaterial interface. The external forces acting on the planar crack surfaces $(S^{\pm})$ are in self-equilibrium. The displacements and the stresses are presumed to decrease to zero at infinity.

The displacement in the upper space I due to the crack disturbance can be expressed in terms of Somigliana's identity (Brebbia, 1981) as

$$u_i^{\mathrm{I}}(\mathbf{x}) = -\int_{S^+} T_{ij}^{\mathrm{I}+}(\mathbf{x}, \xi)\Delta u_j^{\mathrm{I}}(\xi)dS(\xi), \tag{1}$$

in which the superscript I means the upper half-space I; $\Delta u_j^{\mathrm{I}}(\xi)$ are the unknown displacement jumps across the crack surfaces $(S^{\pm})$, i.e.,

$$\Delta u_j^{\mathrm{I}}(\xi) = \{u_j^{\mathrm{I}}(\xi)\}_{x_3=h^+} - \{u_j^{\mathrm{I}}(\xi)\}_{x_3=h^-}$$

and $T_{ij}^{\mathrm{I}+}(\mathbf{x}, \xi)$ denote the tractions in the i direction at a point $\mathbf{x}$ generated by a unit concentrated body force in the j direction

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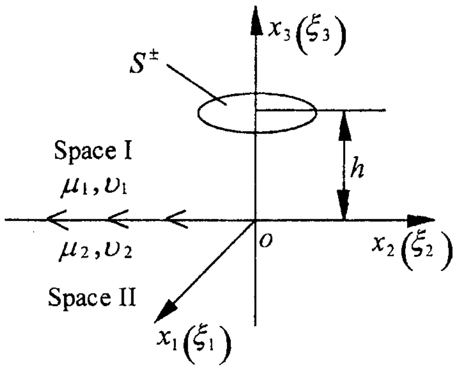

Fig. 1 A planar crack parallel to a bimaterial interface

applied at a point ξ in the upper half-space I (Chen and Tang, 1997).

By using (1) and the generalized Hooke's law, the corresponding stress fields read

$$\sigma_{ij}^{\mathrm{I}}(\mathbf{x}) = -C_{ijkl}^{\mathrm{I}} \int_{S^+} \frac{\partial T_{km}^{\mathrm{I}+}(\mathbf{x}, \xi)}{\partial x_l} \Delta u_m^{\mathrm{I}}(\xi) dS(\xi), \quad (2)$$

where C_{ijkl}^{I} are the components of the elastic modulus tensor in the upper half-space I, given by

$$C_{ijkl}^{\mathrm{I}} = \frac{2\upsilon_1\mu_1}{1 - 2\upsilon_1} \delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where δ_{mn} is the Kronecker delta.

Noting that the differentiations of $T_{ij}^{\mathrm{I}+}(\mathbf{x}, \xi)$ in (2) have hypersingularities of $O(r^3)$ as the point $\mathbf{x}$ approaches the point ξ on the crack surfaces $S^+(x_3 \to h)$ and the boundary conditions over the upper crack surface $\sigma_{3i}^{\mathrm{I}}(x_1, x_2, h^+) = -p_i^{\mathrm{I}}(x_1, x_2, h^+)$, we must employ the concept of the finite-part integral while obtaining the limits of the stress components $\sigma_{3i}^{\mathrm{I}}(x_1, x_2, h^+)$ which give the crack surface tractions.

After lengthy manipulations, a system of two-dimensional hypersingular integral equations for the planar crack parallel to the bimaterial interface is written as

$$\oint_{S^+} \left[\frac{2(\kappa_1 - 1)}{r^3} \delta_{\alpha\beta} + \frac{3(3 - \kappa_1)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{r^5} \right] \Delta u_\beta^{\mathrm{I}}(\xi) dS(\xi) + \int_{S^+} K_{\alpha j}^{\mathrm{I}}(\mathbf{x}, \xi) \Delta u_j^{\mathrm{I}}(\xi) dS(\xi)$$
$$= -\frac{4\pi(\kappa_1 + 1)}{\mu_1} p_\alpha^{\mathrm{I}}(x_1, x_2, h^+), \quad x_1, x_2 \in S^+, \quad (3a)$$

$$\oint_{S^+} \frac{4}{r^3} \Delta u_3^{\mathrm{I}}(\xi) dS(\xi) + \int_{S^+} K_{3j}^{\mathrm{I}}(\mathbf{x}, \xi) \Delta u_j^{\mathrm{I}}(\xi) dS(\xi)$$
$$= -\frac{4\pi(\kappa_1 + 1)}{\mu_1} p_3^{\mathrm{I}}(x_1, x_2, h^+), \quad x_1, x_2 \in S^+, \quad (3b)$$

in which

$$\kappa_1 = 3 - 4\upsilon_1; \quad r^2 = \delta_{\alpha\beta}(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta);$$

$\oint_{S^\pm}$ means the finite-part integral of Hadamard (1952) and the kernels $K_{ij}^{\mathrm{I}}(\mathbf{x}, \xi)$ read

$$K_{11}^{\mathrm{I}}(\mathbf{x}, \xi) = -2[(\kappa_1 - 1) + (\kappa_1 + 1)(\Lambda_1 + \Lambda_2 - 2\Lambda)]/R^3$$
$$+ 3\{4h^2[(\kappa_1 - 5) + 2(\kappa_1 + 1)(3\Lambda_1 - \Lambda)]$$
$$- (x_1 - \xi_1)^2[(3 - \kappa_1) + 2(\kappa_1 + 1)(\Lambda - \Lambda_1 - \Lambda_2)]\}/R^5$$
$$+ 120[1 - (\kappa_1 + 1)\Lambda_1]h^2[4h^2 + 3(x_1 - \xi_1)^2]/R^7$$
$$- 3360[1 - (\kappa_1 + 1)\Lambda_1]h^4(x_1 - \xi_1)^2/R^9,$$

$$K_{21}^{\mathrm{I}}(\mathbf{x}, \xi) = -3(x_1 - \xi_1)(x_2 - \xi_2)\{[(3 - \kappa_1)$$
$$+ 2(\kappa_1 + 1)(\Lambda - \Lambda_1 - \Lambda_2)]/R^5 - 40[1 - (\kappa_1 + 1)\Lambda_1]h^2$$
$$\times (3/R^7 + 28h^2/R^9)\},$$

$$K_{31}^{\mathrm{I}}(\mathbf{x}, \xi) = -12h(x_1 - \xi_1)\{(\kappa_1 + 1)(\Lambda_1 - \Lambda_2)/R^5$$
$$- 20[1 - (\kappa_1 + 1)\Lambda_1]h^2(3/R^7 + 28h^2/R^9)\}$$

$$K_{33}^{\mathrm{I}}(\mathbf{x}, \xi) = -2[2 - (\kappa_1 + 1)(\Lambda_1 + \Lambda_2)]/R^3$$
$$+ 24h^2\{[(\kappa_1 + 1)(2\Lambda_1 - \Lambda_2) - 1]/R^5$$
$$- 80[1 - (\kappa_1 + 1)\Lambda_1]h^2(1/R^7 - 7h^2/R^9)\},$$

$$K_{22}^{\mathrm{I}}(\mathbf{x}, \xi) = K_{11}^{\mathrm{I}}\{x_1 \to x_2, \xi_1 \to \xi_2\},$$
$$K_{12}^{\mathrm{I}}(\mathbf{x}, \xi) = K_{21}^{\mathrm{I}}\{x_1 \leftrightarrow \xi_1, x_2 \leftrightarrow \xi_2\},$$
$$K_{32}^{\mathrm{I}}(\mathbf{x}, \xi) = K_{31}^{\mathrm{I}}\{x_1 \to x_2, \xi_1 \to \xi_2\},$$
$$K_{13}^{\mathrm{I}}(\mathbf{x}, \xi) = K_{31}^{\mathrm{I}}\{x_1 \leftrightarrow \xi_1, x_2 \leftrightarrow \xi_2\},$$
$$K_{23}^{\mathrm{I}}(\mathbf{x}, \xi) = K_{32}^{\mathrm{I}}\{x_1 \leftrightarrow \xi_1, x_2 \leftrightarrow \xi_2\} \quad (4)$$

where

$$\Lambda = \mu_2/(\mu_1 + \mu_2), \quad \Lambda_1 = \mu_2/(\mu_1 + \kappa_1\mu_2),$$
$$\Lambda_2 = \mu_2/(\mu_2 + \kappa_2\mu_1), \quad R^2 = r^2 + 4h^2, \quad \kappa_2 = 3 - 4\upsilon_2.$$

Hypersingular Integro-differential Equation for the Planar Interfacial Crack

It can be seen from (4) that moving the planar crack to the bimaterial interface ($h \to 0$) leads to $R = r$ in the expressions of $K_{ij}^{\mathrm{I}}(\mathbf{x}, \xi)$ and that the kernels $K_{ij}^{\mathrm{I}}(\mathbf{x}, \xi)$ have subsequently hypersingularities of Hadamard (1952) as $\mathbf{x} \to \xi$. To regularize the kernels $K_{ij}^{\mathrm{I}}(\mathbf{x}, \xi)$, $\Delta u_j^{\mathrm{I}}(\xi)$ can be expanded in a two-term Taylor series in the neighborhood of the point $\mathbf{x}$, that is

$$\Delta u_j^{\mathrm{I}}(\xi) = \Delta u_j^{\mathrm{I}}(x_1, x_2) + \frac{\partial \Delta u_j^{\mathrm{I}}(x_1, x_2)}{\partial x_1}(\xi_1 - x_1) + \frac{\partial \Delta u_j^{\mathrm{I}}(x_1, x_2)}{\partial x_2}(\xi_2 - x_2) + O(r^2). \quad (5)$$

Substituting (5) into (3), we have

$$\mu_1(\Lambda_2 - \Lambda_1)\frac{\partial \Delta u_3^{\mathrm{I}}(x_1, x_2)}{\partial x_\alpha} + \mu_1 \frac{(2\Lambda - \Lambda_1 - \Lambda_2)}{2\pi} \times \oint_{S^+} \frac{1}{r^3} \delta_{\alpha\beta} \Delta u_\beta^{\mathrm{I}}(\xi) dS(\xi) + 3\mu_1 \frac{(\Lambda_1 + \Lambda_2 - \Lambda)}{2\pi}$$
$$\times \oint_{S^+} \frac{(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{r^5} \Delta u_\beta^{\mathrm{I}}(\xi) dS(\xi) = -p_\alpha^{\mathrm{I}}(x_1, x_2),$$
$$x_1, x_2 \in S^+, \quad (6a)$$

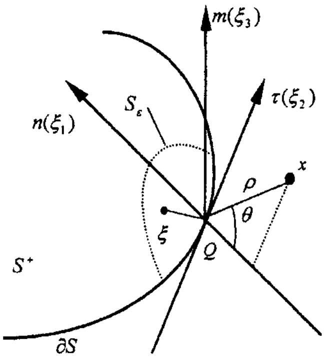

Fig. 2 A local intrinsic coordinate system

$$\mu_1(\Lambda_1 - \Lambda_2)\frac{\partial \Delta u^{\mathrm{I}}_{\alpha}(x_1, x_2)}{\partial x_\alpha} + \mu_1 \frac{(\Lambda_1 + \Lambda_2)}{2\pi} \oint_{S^+} \frac{\Delta u^{\mathrm{I}}_3(\xi)}{r^3} dS(\xi) = -p^{\mathrm{I}}_3(x_1, x_2),$$

$$x_1, x_2 \in S^+. \quad (6b)$$

Equations (6) are the very hypersingular integro-differential equations for the three-dimensional fields of a planar crack on a bimaterial interface. Obviously, if the two dissimilar materials bounded together are the same, that is, $\mu_1 = \mu_2$, $v_1 = v_2$, then $\Lambda = 1/2$, $\Lambda_1 = \Lambda_2$, and (6) degenerate into the results for the infinite homogeneous and isotropic body obtained by Takakuda (1985) and Qin and Tang (1993). We are not aware of any published research on the hypersingular integro-differential equations or other similar integral equations of the three-dimensional interfacial crack fields which are helpful for us to study three-dimensional interfacial crack problems.

Stress Singularity Near the Planar Interfacial Crack Front

As shown in Fig. 2, introduce a local triple orthogonal intrinsic coordinate system: $(n, \tau, m) := (\xi_1, \xi_2, \xi_3)$ at a point Q along the smooth periphery ∂S of the planar crack and a small sector shape of S_ϵ in the osculating plane around the point Q. In terms of the theory of elasticity (Parton and Perlin, 1984), the displacements surrounding the point Q can be decomposed into combinations of those for the plane-strain case with those for the antiplane case, and are related to only coordinates ξ_1 in the normal plane through the point Q as well as the position of the point Q along the contour of the crack, i.e.,

$$\Delta u^{\mathrm{I}}_i(\xi) = A_i(Q)\xi_1^{\alpha_i}. \quad (7)$$

Here the repeated indices are not summed; $A_i(Q)$ are complex constants concerning with the position of the point Q and α_i are unknown singularities which take values within range $0 < \mathrm{Re}(\alpha_i) < 1$.

If bounded internal pressures are applied on the crack surfaces, inserting (7) into (3) and using a similar method to what was previously mentioned, we give by the dominant-part analysis of the two-dimensional hypersingular integrals

$$\lim_{x\to Q} \oint_{S_\epsilon} \frac{1}{r^3} \Delta u^{\mathrm{I}}_m(\xi) dS(\xi) = -2A_m(Q)x_1^{\alpha_m - 1}\alpha_m \pi \cot(\alpha_m \pi), \quad (8a)$$

$$\lim_{x\to Q} \oint_{S_\epsilon} \frac{(x_\beta - \xi_\beta)^2}{r^5} \Delta u^{\mathrm{I}}_\beta(\xi) dS(\xi) = -\frac{2}{3}(2\delta_{\beta 1} + \delta_{\beta 2})A_\beta(Q)x_1^{\alpha_\beta - 1}\alpha_\beta \pi \cot(\alpha_\beta \pi), \quad (8b)$$

$$\lim_{x\to Q}\lim_{h\to 0} \oint_{S_\epsilon} K^{\mathrm{I}}_{nm}(\mathbf{x}, \xi)\Delta u^{\mathrm{I}}_m(\xi) dS(\xi)$$
$$= 2\{2[2 - (\kappa_1 + 1)(\Lambda_1 + \Lambda_2)](\delta_{n1}\delta_{m1} + \delta_{n3}\delta_{m3}) + (\kappa_1 + 1)(1 - 2\Lambda)\delta_{n2}\delta_{m2} + 2(\kappa_1 + 1)(\Lambda_1 - \Lambda_2) \times (\delta_{n3}\delta_{m1} - \delta_{n1}\delta_{m3}) \tan(\alpha_m \pi)\}$$
$$\times A_m(Q)x_1^{\alpha_m - 1}\alpha_m \pi \cot(\alpha_m \pi) \quad (8c)$$

in which the repeated indices are not summed. Substituting (8) into (3), then multiplying both sides of (3) by $x_1^{1-\alpha_m}$ and setting $\alpha_1 = \alpha_3 = \alpha$, yield

$$A_1(Q)(\Lambda_1 + \Lambda_2)\alpha \cot(\pi\alpha) + A_3(Q)(\Lambda_1 - \Lambda_2)\alpha = 0, \quad (9a)$$

$$A_2(Q)\cot(\pi\alpha_2) = 0, \quad (9b)$$

$$A_1(Q)(\Lambda_1 - \Lambda_2)\alpha - A_3(Q)(\Lambda_1 + \Lambda_2)\alpha \cot(\pi\alpha) = 0. \quad (9c)$$

For nontrivial solutions of $A_i(Q)$, the determinant of their coefficients must vanish. Because $\alpha = 0$ is not a solution of above equations in the present case, the following two characteristic equations are obtained:

$$\cot \pi\alpha_2 = 0, \quad (10a)$$

$$\cot^2 \pi\alpha + \left(\frac{\Lambda_1 - \Lambda_2}{\Lambda_1 + \Lambda_2}\right)^2 = 0. \quad (10b)$$

It is seen that (10a) corresponds to antiplane problems (Zak and Williams, 1963) and (10b) to plane-strain problems (Williams, 1959). Therefore, the solutions of (10) read

$$\alpha_2 = \frac{1}{2}; \quad \alpha = \frac{1}{2} \pm i\gamma, \quad \gamma = \frac{1}{2\pi}\ln\left(\frac{\mu_1 + \kappa_1\mu_2}{\mu_2 + \kappa_2\mu_1}\right). \quad (11)$$

Here $i = \sqrt{-1}$. From (9a), (9c), and (10b), we find

$$A_1(Q) = \pm i A_3(Q). \quad (12)$$

As the displacement must be a real number, in terms of (7), (11), and (12), the displacement jumps near the planar interfacial crack front edge can be expressed as

$$\Delta u^{\mathrm{I}}_3(\xi) - i\Delta u^{\mathrm{I}}_1(\xi) = [A_R(Q) + iA_I(Q)]\xi_1^{\alpha} = A(Q)\xi_1^{\alpha}, \quad (13a)$$

$$\Delta u^{\mathrm{I}}_2(\xi) = A_2(Q)\xi_1^{1/2}. \quad (13b)$$

Singular Stress Field Ahead of the Planar Interfacial Crack Front

The stress fields at any point interior to the upper half-space, caused by the planar interfacial crack, have been given by the first author (1997) and take the following form:

$$\sigma^{\mathrm{I}}_{3\alpha}(\mathbf{x}) = \frac{\mu_1}{2\pi}\int_{S^+}\left\{\left[\left(\frac{2\Lambda - \Lambda_1 - \Lambda_2}{r_1^3} + \frac{3(2\Lambda_1 - \Lambda)}{r_1^5}x_3^2\right)\delta_{\alpha\beta} + \left(\frac{3(\Lambda_1 + \Lambda_2 - \Lambda)}{r_1^5} - \frac{30\Lambda_1}{r_1^7}x_3^2\right)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)\right]\Delta u^{\mathrm{I}}_\beta(\xi) + \left(\frac{3(3\Lambda_1 - \Lambda_2)}{r_1^5} - \frac{30\Lambda_1}{r_1^7}x_3^2\right)x_3(x_\alpha - \xi_\alpha)\Delta u^{\mathrm{I}}_3(\xi)\right\}dS(\xi),$$

$$(14a)$$

$$\sigma_{33}^{\mathrm{I}}(\mathbf{x}) = \frac{\mu_1}{2\pi}\int_{S^+}\left\{\left(\frac{3(\Lambda_1+\Lambda_2)}{r_1^5} - \frac{30\Lambda_1}{r_1^7}x_3^2\right)\right.$$

$$\times x_3(x_\beta - \xi_\beta)\Delta u_\beta^{\mathrm{I}}(\boldsymbol{\xi}) + \left(\frac{(\Lambda_1+\Lambda_2)}{r_1^3} + \frac{3(5\Lambda_1-\Lambda_2)}{r_1^5}\right.$$

$$\left.\left.\times x_3^2 - \frac{30\Lambda_1}{r_1^7}x_3^4\right)\Delta u_3^{\mathrm{I}}(\boldsymbol{\xi})\right\}dS(\boldsymbol{\xi}), \quad (14b)$$

where

$$r_1^2 = (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2.$$

To get the singular stress fields ahead of the periphery of the planar interfacial crack we use a method similar to that of Qin and Tang (1993). As shown in Fig. 2, introducing a polar coordinate system (ρ, θ) referring to the negative normal direction in the normal plane (Q, n, m) through the point Q leads to the following coordinates at the point $\mathbf{x}$ ahead of the crack front:

$$x_1 = -\rho\cos\theta, \quad x_2 = 0, \quad x_3 = \rho\sin\theta, \quad (15a)$$

$$r_1^2 = (\xi_1 + \rho\cos\theta)^2 + \xi_2^2 + (\rho\sin\theta)^2. \quad (15b)$$

Substituting (13) and (15) into (14), we obtain dominant-part integrals as follows:

$$\lim_{\mathbf{x}\to Q}\int_{S_\epsilon}\frac{\Delta u_m^{\mathrm{I}}(\boldsymbol{\xi})}{r_1^3}dS(\boldsymbol{\xi}) = \frac{2\pi}{\rho\sin\theta}\left\{-\delta_{m1}\,\mathrm{Re}\left[iA(Q)\rho^\alpha\frac{\sin\alpha\theta}{\sin\alpha\pi}\right]\right.$$

$$\left.+\delta_{m3}\,\mathrm{Im}\left[iA(Q)\rho^\alpha\frac{\sin\alpha\theta}{\sin\alpha\pi}\right]\right\}, \quad (16a)$$

$$\lim_{\mathbf{x}\to Q}\int_{S_\epsilon}\frac{(x_1-\xi_1)\Delta u_m^{\mathrm{I}}(\boldsymbol{\xi})}{r_1^5}dS(\boldsymbol{\xi})$$

$$= \frac{2\pi}{3\rho\sin\theta}\left\{\delta_{m1}\,\mathrm{Re}\left[iA(Q)\alpha\rho^{\alpha-1}\frac{\sin(\alpha-1)\theta}{\sin\alpha\pi}\right]\right.$$

$$\left.-\delta_{m3}\,\mathrm{Im}\left[iA(Q)\alpha\rho^{\alpha-1}\frac{\sin(\alpha-1)\theta}{\sin\alpha\pi}\right]\right\}, \quad (16b)$$

$$\lim_{\mathbf{x}\to Q}\int_{S_\epsilon}\frac{(x_1-\xi_1)(x_2-\xi_2)\Delta u_m^{\mathrm{I}}(\boldsymbol{\xi})}{r_1^5}dS(\boldsymbol{\xi}) = 0, \quad (16c)$$

$$\lim_{\mathbf{x}\to Q}\int_{S_\epsilon}\frac{(x_1-\xi_1)^2\Delta u_m^{\mathrm{I}}(\boldsymbol{\xi})}{r_1^5}dS(\boldsymbol{\xi})$$

$$= \frac{2\pi}{3}\left\{\delta_{m1}\left\{\mathrm{Im}\left[iA(Q)\alpha\rho^{\alpha-1}\frac{\sin(\alpha-1)\theta}{\sin\alpha\pi}\right]\right.\right.$$

$$\left.-\frac{1}{\rho\sin\theta}\mathrm{Re}\left[iA(Q)\rho^\alpha\frac{\sin\alpha\theta}{\sin\alpha\pi}\right]\right\}$$

$$+\delta_{m3}\left\{\mathrm{Im}\left[iA(Q)\alpha\rho^{\alpha-1}\frac{\sin(\alpha-1)\theta}{\sin\alpha\pi}\right]\right.$$

$$\left.\left.+\frac{1}{\rho\sin\theta}\mathrm{Re}\left[iA(Q)\rho^\alpha\frac{\sin\alpha\theta}{\sin\alpha\pi}\right]\right\}\right\}, \quad (16d)$$

in which $\alpha = 1/2 + i\gamma$. Putting the above results into (14), we arrive at

$$\sigma_{31}^{\mathrm{I}}(\rho,\theta) = \frac{\mu_1\Lambda_1 e^{\pi\gamma}}{2\cosh\gamma\pi}\cdot\frac{1}{\sqrt{\rho}}\{[(A_R - 2\gamma A_I)\cos(\gamma\ln\rho)$$

$$-(A_I + 2\gamma A_R)\sin(\gamma\ln\rho)]\sigma_{31}^1(\theta) - [(A_I + 2\gamma A_R)$$

$$\times\cos(\gamma\ln\rho) + (A_R - 2\gamma A_I)\sin(\gamma\ln\rho)]\sigma_{31}^2(\theta)\}, \quad (17a)$$

$$\sigma_{33}^{\mathrm{I}}(\rho,\theta) = \frac{\mu_1\Lambda_1 e^{\pi\gamma}}{2\cosh\gamma\pi}\cdot\frac{1}{\sqrt{\rho}}\{[(A_R - 2\gamma A_I)\cos(\gamma\ln\rho)$$

$$-(A_I + 2\gamma A_R)\sin(\gamma\ln\rho)]\sigma_{33}^2(\theta) + [(A_I + 2\gamma A_R)$$

$$\times\cos(\gamma\ln\rho) + (A_R - 2\gamma A_I)\sin(\gamma\ln\rho)]\sigma_{33}^1(\theta)\}, \quad (17b)$$

$$\sigma_{32}^{\mathrm{I}}(\rho,\theta) = \frac{\mu_1\Lambda}{2}A_2(Q)\frac{1}{\sqrt{\rho}}\cos\frac{\theta}{2}, \quad (17c)$$

where $\sigma_{3m}^1(\theta)$ and $\sigma_{3m}^2(\theta)$ $(m = 1, 3)$ are nondimensional angular functions of the singular stress fields $\sigma_{3m}^{\mathrm{I}}(\rho,\theta)$. They are expressed as

$$\sigma_{3m}^1(\theta) = 2\sin\frac{\theta}{2}\sinh(\pi-\theta)\gamma$$

$$+(1-2\delta_{m1})e^{-(\pi-\theta)\gamma}\sin\theta\left(\cos\frac{3\theta}{2} - 2\gamma\sin\frac{3\theta}{2}\right),$$

$$\sigma_{3m}^2(\theta) = 2\cos\frac{\theta}{2}\cosh(\pi-\theta)\gamma$$

$$+(1-2\delta_{m1})e^{-(\pi-\theta)\gamma}\sin\theta\left(\sin\frac{3\theta}{2} + 2\gamma\cos\frac{3\theta}{2}\right).$$

Equations (17) are the very singular stress fields ahead of the planar interfacial crack front edge in the upper half-space I. It is shown that $\sigma_{31}^{\mathrm{I}}(\rho,\theta)$ and $\sigma_{33}^{\mathrm{I}}(\rho,\theta)$ are oscillatory and not of a separable form. In contrast, the singular stress fields of cracks in homogenous media are separable and not oscillating (Tang and Qin, 1993). This conclusion is the same as that of plane crack problems but it is drawn strictly from the three-dimensional elasticity theory.

Stress Intensity Factor and Energy Release Rate for the Planar Interfacial Crack

Analogous to Hutchinson's definitions (Hutchinson et al., 1987) of stress intensity factors of the two-dimensional interfacial crack tip, we define the stress intensity factors associated with the three-dimensional planar interfacial crack front edge as follows:

$$K(Q) = K_1(Q) - iK_2(Q)$$

$$= \lim_{\rho\to 0}\sqrt{2}\rho^{1/2-i\gamma}[\sigma_{33}^{\mathrm{I}}(\rho,\theta) - i\sigma_{31}^{\mathrm{I}}(\rho,\theta)]_{\theta=0}$$

$$= \lim_{\rho\to 0}\sqrt{2}\rho^{1/2-i\gamma}[\sigma_{33}^{\mathrm{I}}(\rho,0) - i\sigma_{31}^{\mathrm{I}}(\rho,0)], \quad (18a)$$

$$K_{\mathrm{III}}(Q) = \lim_{\rho\to 0}\sqrt{2\rho}\,\sigma_{32}^{\mathrm{I}}(\rho,\theta)|_{\theta=0} = \lim_{\rho\to 0}\sqrt{2\rho}\,\sigma_{32}^{\mathrm{I}}(\rho,0). \quad (18b)$$

From (13), (17), and (18) we obtain

$$K(Q) = \frac{\mu_1(\Lambda_1+\Lambda_2)(1+2i\gamma)}{2\cosh\pi\gamma}$$

$$\times\lim_{\xi_1\to 0}[\Delta u_3^{\mathrm{I}}(\xi_1,0) - i\Delta u_1^{\mathrm{I}}(\xi_1,0)]\sqrt{\frac{2}{\xi_1}}(\xi_1)^{-i\gamma}$$

$$= \frac{\mu_1\mu_2(1+2i\gamma)}{2\cosh\pi\gamma}\left(\frac{1}{\mu_1+\kappa_1\mu_2} + \frac{1}{\mu_2+\kappa_2\mu_1}\right)$$

$$\times\lim_{\xi_1\to 0}[\Delta u_3^{\mathrm{I}}(\xi_1,0) - i\Delta u_1^{\mathrm{I}}(\xi_1,0)]$$

$$\times\sqrt{\frac{2}{\xi_1}}(\xi_1)^{-i\gamma}, \quad (19a)$$

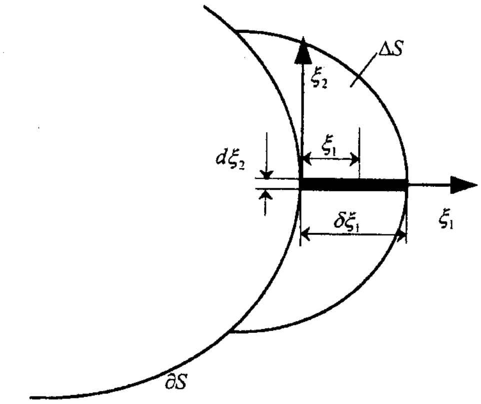

Fig. 3 A planar crack with virtual incremental area ΔS of the crack surface

$$K_{\rm III}(Q) = \frac{\mu_1 \Lambda}{\sqrt{2}} \lim_{\xi_1 \to 0} \frac{\Delta u_2^{\rm I}(\xi_1, 0)}{\sqrt{\xi_1}}$$

$$= \frac{\mu_1 \mu_2}{2(\mu_1 + \mu_2)} \lim_{\xi_1 \to 0} \Delta u_2^{\rm I}(\xi_1, 0) \sqrt{\frac{2}{\xi_1}}. \quad (19b)$$

To the authors' knowledge, the present derivation of (17) and (19) is the first to do so rigorously from the three-dimensional elasticity theory.

From (19), the displacement jumps across the crack surfaces near the interfacial crack front edge take the form

$$\Delta U(\xi_1, 0) = \Delta u_3^{\rm I}(\xi_1, 0) - i \Delta u_1^{\rm I}(\xi_1, 0)$$

$$= \frac{K(Q)}{2(1 + 2i\gamma) \cosh \pi\gamma} \left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2} \right) \times \sqrt{\frac{\xi_1}{2}} (\xi_1)^{i\gamma}, \quad (20a)$$

$$\Delta u_2^{\rm I}(\xi_1, 0) = 2 K_{\rm III}(Q) \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \sqrt{\frac{\xi_1}{2}}. \quad (20b)$$

By means of virtual work, the given expressions for tractions (18) and displacement fields (20), the energy release rate is readily computed from

$$G(Q) = \lim_{\Delta S \to 0} \frac{1}{2\Delta S} \int_{\Delta S} \{[\sigma_{33}^{\rm I}(\xi_1; 0) - i\sigma_{31}^{\rm I}(\xi_1; 0)]$$

$$\cdot \Delta \bar{U}(\xi_1, 0 + \Delta S) + \sigma_{32}^{\rm I}(\xi_1; 0) \Delta u_2^{\rm I}(\xi_1; 0 + \Delta S)\} dS(\xi), \quad (21)$$

where $\sigma_{3i}^{\rm I}(\xi_1, 0)$ are the stress components ahead of the planar interfacial crack front edge, when the increment ΔS of the crack is zero; $\Delta \bar{U}(\xi_1; 0 + \Delta S)$ is the complex conjugate to the displacement jump vector $\Delta U(\xi_1; 0 + \Delta S)$, and $\Delta u_2^{\rm I}(\xi_1; 0 + \Delta S)$, the displacement jump, when the crack has advanced to the position $0 + \Delta S$. The infinitesimal amount $\delta\xi_1$ normal to the planar interfacial crack front edge ∂S is used to describe the changes of the crack surface (see Fig. 3). The total changes of the crack surface are expressed as

$$\Delta S = \int_{\partial S(\xi_2)} \delta\xi_1 d\xi_2. \quad (22)$$

Inserting (18), (20), and (22) into (21) gives

$$G(Q) = \lim_{\Delta S \to 0} \frac{1}{2 \displaystyle\int_{\partial S(\xi_2)} \delta\xi_1 d\xi_2}$$

$$\times \int_{\partial S(\xi_2)} \int_0^{\delta\xi_1} \left[\frac{K(Q)\overline{K(Q)}}{4(1 - 2i\gamma) \cosh \pi\gamma} \right.$$

$$\times \left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2} \right) \left(\frac{\delta\xi_1 - \xi_1}{\xi_1} \right)^{1/2 - i\gamma}$$

$$\left. + \frac{(\mu_1 + \mu_2) K_{\rm III}^2(Q)}{\mu_1 \mu_2} \sqrt{\frac{\delta\xi_1 - \xi_1}{\xi_1}} \right] d\xi_1 d\xi_2 \quad (23)$$

where the inner integral of the first term is recognizable as the complex beta function $B(1/2 + i\gamma, 3/2 - i\gamma)$, and the inner integral of the second term, the real beta function $B(1/2, 3/2)$. Upon evaluation of the beta function by the gamma function $\Gamma(z)$,

$$G(Q) = \pi \left[\frac{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)}{16 \mu_1 \mu_2 \cosh^2 \pi\gamma} \right] K(Q)\overline{K(Q)}$$

$$+ \pi \frac{(\mu_1 + \mu_2)}{4 \mu_1 \mu_2} K_{\rm III}^2(Q). \quad (24)$$

Equation (24) notes that the integral in (21) has no imaginary part, a fact noticed in the general case of the two-dimensional bimaterials by Shih and Asaro (1988).

Illustrative Example

The theoretical studies in the previous sections have shown that the stress fields ahead of the planar interfacial crack have a singularity of $1/\sqrt{r} \times$ (oscillatory term) which complicates our analyses of crack problems. However, Rice (1988) and Shibuya (1989) pointed out that the phenomenon of stress oscillation occurs only in the extreme vicinity of the crack tip under tensile loading. Therefore, to reduce the efforts of our computations, we here ignore the oscillatory term in the process of numerical calculations.

As an illustration, we consider an elliptical crack under normal constant pressures. The elliptical crack with semi-major axis a and semi-minor axis b is represented as

$$\xi_1 = a\rho \cos \varphi, \quad \xi_2 = b\rho \sin \varphi,$$

where $0 \le \rho < 1$, $0 \le \varphi < 2\pi$.

In terms of the behavior of the displacement jumps over the

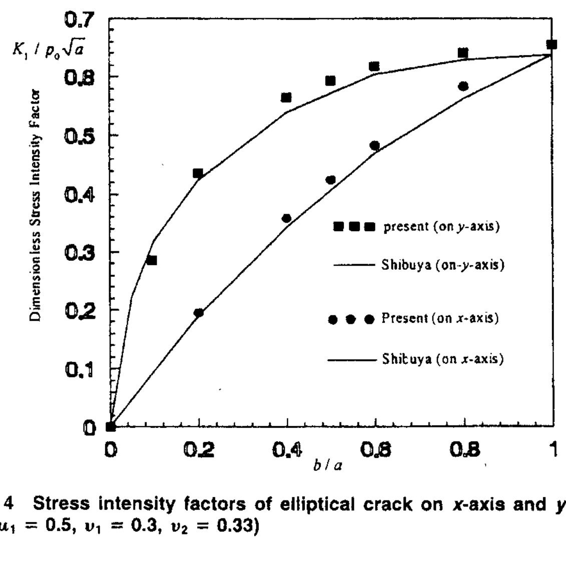

Fig. 4 Stress intensity factors of elliptical crack on x-axis and y-axis ($\mu_2/\mu_1 = 0.5$, $\nu_1 = 0.3$, $\nu_2 = 0.33$)

crack surfaces without consideration of the oscillatory term, the crack displacement jumps can be expressed in the form of Chebyshev polynomial, i.e.,

$$\Delta u_i^I(\xi_1, \xi_2) = \sqrt{1 - (\xi_1/a)^2 - (\xi_2/b)^2} \times \sum_{n=0,2,\ldots}^{N} \sum_{m=0,2,\ldots}^{M} C_{nm}^i T_n(\xi_1/a) T_m(\xi_2/b), \quad (25)$$

in which C_{nm}^i are unknown coefficients to be determined and T_l denotes the Chebyshev polynomials of the first kind. Consequently, the mode I stress intensity factor can be obtained from the following expression (Chen, 1997):

$$K_I(Q) = \mu_1\mu_2\left(\frac{1}{\mu_1 + \kappa_1\mu_2} + \frac{1}{\mu_2 + \kappa_2\mu_1}\right) \times \sum_{n=0,2,\ldots}^{N} \sum_{m=0,2,\ldots}^{M} C_{nm}^3 T_n(\cos\varphi) T_m(\sin\varphi) \sqrt{\frac{1}{ab}}\,\bar{K}, \quad (26)$$

here

$$\bar{K} = (a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{1/4}.$$

The calculated results on the x-axis and y-axis are depicted in Fig. 4 as a function of the aspect ratio of the elliptical crack and compared with the available solutions (Shibuya, 1989). It is seen that both results are in good agreement.

Concluding Remark

In the present paper, starting from the fundamental solution of a point force for three-dimensional elastic space problems of bimaterials, we have performed rigorous theoretical analyses of some basic interfacial crack problems by using the hypersingular integral equation method and have obtained some useful results such as a system of hypersingular integral equations for an arbitrarily shaped planar interfacial crack (6), singular stress fields (17), and stress intensity factors (18). These results are undoubtedly helpful for us to study the three-dimensional interfacial crack problems.

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