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# Analysis of an elliptical crack parallel to a bimaterial interface under tension

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**Abstract**

In this paper an elliptical crack parallel to a bimaterial interface is considered. The solution utilizes the body force method and requires Green's functions for perfectly bonded elastic half planes. The formulation leads to a system of hypersingular integral equations whose unknowns are three modes of crack opening displacements. In the numerical calculation, fundamental density functions and polynomials are used to approximate unknown body force densities. The results show that the present method yields smooth variations of stress intensity factors along the crack front accurately. The stress intensity factors are indicated in tables and figures with varying the shape of crack, distance from the interface, and elastic constants. The root area parameter proposed by Murakami is found to be effective for engineering use because different shaped cracks have almost the same values.


*Keywords:* Elasticity; Crack; Bimaterial; Interface; Elliptical crack; Stress intensity factor; Body force method

## 1. Introduction

With increasing the use of composite materials in engineering structure, much attention has been paid to the strength of interface. Although a lot of researches have been made in terms of fracture mechanics approach regarding interface, most of them generally involve two-dimensional modeling (Erdogan and Aksogan, 1974; Cook and Erdogan, 1972; Isida and Noguchi, 1983). Little work has been carried out on the three-dimensional aspect of crack problems except those of specially shaped cracks (Willis, 1972; Erdogan and Arin, 1972; Kassir and Bregman, 1972; Shibuya et al., 1989; Nakamura, 1991; Yuuki and Xu, 1992). This is mainly due to the extreme difficulties of solving such problems by mathematics and mechanics, or to the substantial computation required in the numerical analyses.

This paper deals with a three-dimensional elliptical crack parallel to an interface as shown in Fig. 1. Previously, only the limiting cases as $a/b \to \infty$ were treated by Isida and Noguchi as a two-dimensional solution (1983). Although in the previous study (Chen et al., 1999) the problem was

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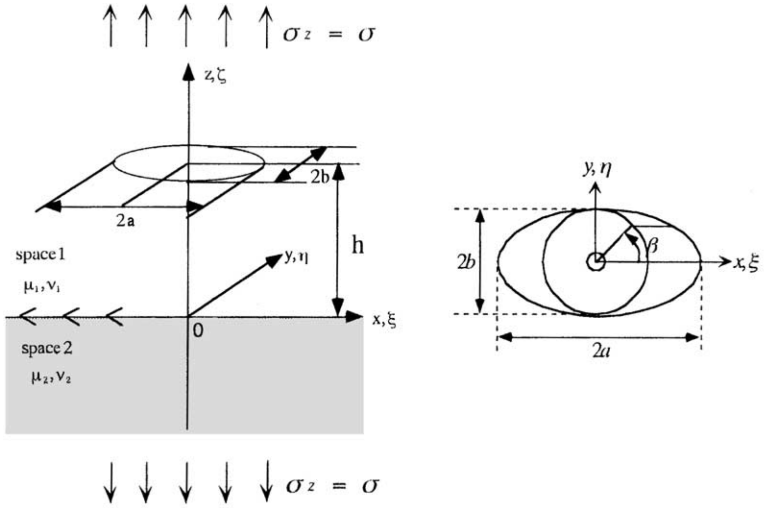



Fig. 1. An elliptical crack parallel to a bimaterial interface ($x^2/a^2 + y^2/b^2 = 1$, $z = h$).

formulated as a system of singular integral equations, there is no numerical solution indicated in tables and figures. In this study the equations will be solved accurately by using fundamental densities and polynomials to approximate unknown functions (Noda and Miyoshi, 1996). Here, the fundamental densities are chosen to express the stress fields due to an elliptical crack in an infinite body exactly. Then, the stress intensity factors will be indicated with varying the shape of crack, elastic constants of materials, and the distance between the crack and interface.

## 2. Singular integral equation of the body force method for a mixed mode surface crack

Consider an elliptical crack parallel to a bimaterial interface, under uniform tension $\sigma_z^\infty$ at infinity as shown in Fig. 1(a). Here, the elliptical crack has principal diameters $2a$ and $2b$. The body force method is used to formulate the problem as a system of singular integral equations, whose unknowns are body force densities $f_{zz}(\xi,\eta)$, $f_{yz}(\xi,\eta)$, $f_{zx}(\xi,\eta)$. The body force densities are equivalent to crack opening displacements as shown in Eq. (1e). Here, $(\xi,\eta,\zeta)$ is a $(x,y,z)$ coordinate where the body force is applied.

$$\frac{(1-2v)}{8\pi(1-v)^2}\left[\oint\!\!\oint_S \frac{f_{zz}(\xi,\eta)}{r^3}\,d\xi\,d\eta + \iint_S \frac{1}{4}K_{zz}^{f_{zz}}(\xi,\eta,x,y)f_{zz}(\xi,\eta)\,d\xi\,d\eta\right] + \frac{1}{8\pi(1-v)}\left[\iint_S \frac{1}{2}K_{zz}^{f_{yz}}(\xi,\eta,x,y)f_{yz}(\xi,\eta)\,d\xi\,d\eta + \iint_S \frac{1}{2}K_{zz}^{f_{zx}}(\xi,\eta,x,y)f_{zx}(\xi,\eta)\,d\xi\,d\eta\right] = -\sigma_z^\infty \tag{1a}$$

$$\frac{1}{8\pi(1-v)}\left[\oint\!\!\oint_S\left\{\frac{2(1-2v)}{r^3}+\frac{6v(y-\eta)^2}{r^5}\right\}f_{yz}(\xi,\eta)\,d\xi\,d\eta + \oint\!\!\oint_S \frac{6v(x-\xi)(y-\eta)}{r^5}f_{zx}(\xi,\eta)\,d\xi\,d\eta + \iint_S \frac{(1-2v)}{4(1-v)}K_{yz}^{f_{zz}}(\xi,\eta,x,y)f_{zz}(\xi,\eta)\,d\xi\,d\eta + \iint_S \frac{1}{2}K_{yz}^{f_{yz}}(\xi,\eta,x,y)f_{yz}(\xi,\eta)\,d\xi\,d\eta + \iint_S \frac{1}{2}K_{yz}^{f_{zx}}(\xi,\eta,x,y)f_{zx}(\xi,\eta)\,d\xi\,d\eta\right] = 0 \tag{1b}$$

$$\frac{1}{8\pi(1-v)}\left[\oint\!\!\!\oint_S \frac{6v(x-\xi)(y-\eta)}{r^5} f_{yz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right.$$
$$+\oint\!\!\!\oint_S \left\{\frac{2(1-2v)}{r^3}+\frac{6v(x-\xi)^2}{r^5}\right\} f_{zx}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\int\!\!\int_S \frac{1-2v}{4(1-v)} K_{zx}^{f_{zz}}(\xi,\eta,x,y) f_{zz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\int\!\!\int_S \frac{1}{2} K_{zx}^{f_{yz}}(\xi,\eta,x,y) f_{yz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\left.+\int\!\!\int_S \frac{1}{2} K_{zx}^{f_{zx}}(\xi,\eta,x,y) f_{zx}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]=0 \quad (1c)$$

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2}$$
$$S=\left\{(\xi,\eta)\middle|(\xi/a)^2+(\eta/b)^2\leqslant 1\right\} \quad (1d)$$

$$U_z(x,y)=u_z(x,y+0)-u_z(x,y-0)=\frac{(1-2v)(1+v)}{E(1-v)} f_{zz}(x,y)$$
$$U_y(x,y)=u_y(x,y+0)-u_y(x,y-0)=\frac{2(1+v)}{E} f_{yz}(x,y) \quad (1e)$$
$$U_x(x,y)=u_x(x,y+0)-u_x(x,y-0)=\frac{2(1+v)}{E} f_{zx}(x,y)$$

Eqs. (1a)–(1c) enforce boundary conditions at the prospective boundary $S$ for crack; that is, $\sigma_z=0$, $\tau_{yz}=0$, $\tau_{zx}=0$. Eq. (1) includes singular terms in the form of $1/r_1^3$, $1/r_1^5$ corresponding to the ones of an elliptical crack in an infinite body. Therefore the integration $\oint\!\!\oint_S$ should be interpreted in a sense of a finite part integral (Hadamard, 1923) in the region $S$. On the other hand, the integral $\int\!\int_S$ does not include singular terms. As an example, the notation $K_{zz}^{f_{zz}}(\xi,\eta,x,y)$ refers to a function that satisfies the boundary condition for free surface. Correct equations are shown in (2) because of some misprints in the previous paper (Chen et al., 1999).

$$\begin{aligned}K_{zx}^{f_{zx}}(\xi,\eta,x,y)&=-2[(\kappa_1-1)+(\kappa_1+1)(\Lambda_1+\Lambda_2-2\Lambda)]/R^3\\&\quad+3\{4h^2[(\kappa_1-5)+2(\kappa_1+1)(3\Lambda_1-\Lambda_2)]\\&\quad-(x-\xi)^2[(3-\kappa_1)+2(\kappa_1+1)(\Lambda-\Lambda_1-\Lambda_2)]\}/R^5\\&\quad+120[1-(\kappa_1+1)\Lambda_1]h^2[4h^2+3(x-\xi)^2]/R^7\\&\quad-3360[1-(\kappa_1+1)\Lambda_1]h^4(x-\xi)^2/R^9,\end{aligned} \quad (2a)$$

$$\begin{aligned}K_{yz}^{f_{zx}}(\xi,\eta,x,y)&=-3(x-\xi)(y-\eta)\{[(3-\kappa_1)\\&\quad+2(\kappa_1+1)(\Lambda-\Lambda_1-\Lambda_2)]/R^5\\&\quad-40[1-(\kappa_1+1)\Lambda_1]h^2(3/R^7-28h^2/R^9)\},\end{aligned} \quad (2b)$$

$$\begin{aligned}K_{zz}^{f_{zx}}(\xi,\eta,x,y)&=-12h(x-\xi)\{(\kappa_1+1)(\Lambda_1-\Lambda_2)/R^5\\&\quad-20[1-(\kappa_1+1)\Lambda_1]h^2(3/R^7-28h^2/R^9)\},\end{aligned} \quad (2c)$$

$$\begin{aligned}K_{zz}^{f_{zz}}(\xi,\eta,x,y)&=-2[2-(\kappa_1+1)(\Lambda_1+\Lambda_2)]/R^3\\&\quad+24h^2\{[(\kappa_1+1)(2\Lambda_1-\Lambda_2)-1]/R^5\\&\quad-80[1-(\kappa_1+1)\Lambda_1]h^2(1/R^7-7h^2/R^9)\},\end{aligned} \quad (2d)$$

$$K_{yz}^{f_{yz}}(\xi,\eta,x,y)=K_{zx}^{f_{zx}}\{x\to y,\xi\to\eta\} \quad (2e)$$

$$K_{zx}^{f_{yz}}(\xi,\eta,x,y)=K_{yz}^{f_{zx}}\{x\leftrightarrow\xi,y\leftrightarrow\eta\} \quad (2f)$$

$$K_{zz}^{f_{yz}}(\xi,\eta,x,y)=K_{zz}^{f_{zx}}\{x\to y,\xi\to\eta\} \quad (2g)$$

$$K_{zx}^{f_{zz}}(\xi,\eta,x,y)=K_{zz}^{f_{zx}}\{x\leftrightarrow\xi,y\leftrightarrow\eta\} \quad (2h)$$

$$K_{yz}^{f_{zz}}(\xi,\eta,x,y)=K_{zz}^{f_{yz}}\{x\leftrightarrow\xi,y\leftrightarrow\eta\} \quad (2i)$$

In Eqs. (2e)–(2i), the notation $x\to y$ represents that $x$ should be replaced by $y$. On the other hand, the one $x\leftrightarrow\xi$ represents that $x$ should be replaced by $y$, and $y$ should be replaced by $x$. Also, we have

$$\begin{aligned}&\Lambda=\mu_2/(\mu_1+\mu_2),\\&\Lambda_1=\mu_2/(\mu_1+\kappa_1\mu_2),\\&\Lambda_2=\mu_2/(\mu_2+\kappa_2\mu_1),\quad R^2=r^2+4h^2,\\&\kappa_1=3-4v_1,\quad \kappa_2=3-4v_2\end{aligned} \quad (2j)$$

## 3. Numerical solution of singular integral equations

In the conventional body force method, the crack region is divided into several elements; then, fundamental density functions and step functions were used to approximate unknown functions. However, the use of step functions gives rise to singularities along the element boundaries, and it tends to deteriorate the accuracy and validity in sophisticated problems. In the present analysis, the following expressions have been used to approximate the unknown functions as continuous functions. First, we put

$$
\begin{aligned}
&f_{zz}(\xi,\eta) = F_{zz}(\xi_a,\eta_b)w_{zz}(\xi_a,\eta_b)\\
&f_{yz}(\xi,\eta) = F_{yz}(\xi_a,\eta_b)w_{yz}(\xi_a,\eta_b)\\
&f_{zx}(\xi,\eta) = F_{zx}(\xi_a,\eta_b)w_{zx}(\xi_a,\eta_b)\\
&w_{zz}(\xi_a,\eta_b) = \frac{4(1-\nu)^2 b\sigma_z^\infty}{(1-2\nu)E(k)}\sqrt{1-\xi_a^2-\eta_b^2}\\
&w_{yz}(\xi_a,\eta_b) = \frac{2b(1-\nu)k^2\tau_{yz}^\infty}{C(k)}\sqrt{1-\xi_a^2-\eta_b^2}\\
&w_{zx}(\xi_a,\eta_b) = \frac{2b(1-\nu)k^2\tau_{zx}^\infty}{B(k)}\sqrt{1-\xi_a^2-\eta_b^2}\\
&B(k) = (k^2-\nu)E(k) + \nu k'^2K(k)\\
&C(k) = (k^2+\nu k'^2)E(k) - \nu k'^2K(k)\\
&k' = b/a \leqslant 1 \quad k = \sqrt{1-(b/a)^2}\\
&\xi_a = \xi/a \quad \eta_b = \eta/b\\
&K(k) = \int_0^{\pi/2}\frac{d\lambda}{\sqrt{1-k^2\sin^2\lambda}}\\
&E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\lambda}\,d\lambda
\end{aligned}
\tag{3}
$$

Here, $w_{zz}(\xi_a,\eta_b)$, $w_{yz}(\xi_a,\eta_b)$, $w_{zx}(\xi_a,\eta_b)$ are called a fundamental density function of body force, which express the stress field due to an elliptical crack in an infinite body under uniform tension and shears $\sigma_z^\infty$, $\tau_{yz}^\infty$, $\tau_{zx}^\infty$ and lead to solutions with high accuracy. In this calculation, we put $\sigma_z^\infty = \tau_{yz}^\infty = \tau_{zx}^\infty = 1$. Using the expression (3), Eq. (1) is reduced to Eq. (4), where unknowns are $F_{zz}(\xi_a,\eta_b)$, $F_{yz}(\xi_a,\eta_b)$, $F_{zx}(\xi_a,\eta_b)$, which are called weight functions. The unknown functions are related to

$$
\begin{aligned}
&\frac{b}{2\pi E(k)}\left[\oiint_S \frac{F_{zz}(\xi_a,\eta_b)}{r^3}\sqrt{1-\xi_a^2-\eta_b^2}\,d\xi\,d\eta\right.\\
&\quad\left.+\iint_S \frac{1}{4}K_{zz}^{f_{zz}}(\xi,\eta,x,y)F_{zz}(\xi_a,\eta_b)\sqrt{1-\xi_a^2-\eta_b^2}\,d\xi\,d\eta\right]\\
&\quad+\frac{bk^2}{8\pi}\left[\frac{1}{C(k)}\iint_S K_{zz}^{f_{yz}}(\xi,\eta,x,y)F_{zy}(\xi_a,\eta_b)\right.\\
&\quad\times\sqrt{1-\xi_a^2-\eta_b^2}\,d\xi\,d\eta\\
&\quad+\frac{1}{B(k)}\iint_S K_{zz}^{f_{zx}}(\xi,\eta,x,y)F_{zx}(\xi_a,\eta_b)\\
&\quad\left.\times\sqrt{1-\xi_a^2-\eta_b^2}\,d\xi\,d\eta\right] = -\sigma_z^\infty
\end{aligned}
\tag{4}
$$

Since the problem is symmetric with respect to the $x$- and $y$-axis, the following expressions (5) can be applied to approximate three unknown functions $F_{zz}(\xi_a,\eta_b)$, $F_{yz}(\xi_a,\eta_b)$, $F_{zx}(\xi_a,\eta_b)$. Here, $F_{zz}(\xi_a,\eta_b)$ is an even function of both $\xi_a$ and $\eta_b$, $F_{yz}(\xi_a,\eta_b)$ is even of $\xi_a$ but odd of $\eta_b$, and $F_{zx}(\xi_a,\eta_b)$ is odd of $\xi_a$ but even of $\eta_b$.

$$
\begin{aligned}
&F_{zz}(\xi_a,\eta_b)\\
&\quad= \alpha_0\xi_a^0\eta_b^0 + \alpha_1\xi_a^0\eta_b^2 + \cdots + \alpha_{n-1}\xi_a^0\eta_b^{2\cdot(n-1)} + \alpha_n\xi_a^0\eta_b^{2\cdot n}\\
&\qquad+ \alpha_{n+1}\xi_a^2\eta_b^0 + \alpha_{n+2}\xi_a^2\eta_b^2 + \cdots + \alpha_{2n-1}\xi_a^2\eta_b^{2\cdot(n-1)}\\
&\qquad\quad\vdots \qquad \vdots\\
&\qquad+ \alpha_{l-n-1}\xi_a^{2\cdot n}\eta_b^0\\
&\quad= \sum_{i=0}^{l-1}\alpha_i G_i(\xi_a,\eta_b)
\end{aligned}
\tag{5a}
$$

$$
\begin{aligned}
&F_{yz}(\xi_a,\eta_b)\\
&\quad= \beta_0\xi_a^0\eta_b + \beta_1\xi_a^0\eta_b^3 + \cdots + \beta_{n-1}\xi_a^0\eta_b^{2\cdot n-1} + \beta_n\xi_a^0\eta_b^{2\cdot n+1}\\
&\qquad+ \beta_{n+1}\xi_a^2\eta_b + \beta_{n+2}\xi_a^2\eta_b^3 + \cdots + \beta_{2n-1}\xi_a^2\eta_b^{2\cdot n-1}\\
&\qquad\quad\vdots \qquad \vdots\\
&\qquad+ \beta_{l-n-1}\xi_a^{2\cdot n}\eta_b^0\\
&\quad= \sum_{i=0}^{l-1}\beta_i Q_i(\xi_a,\eta_b)
\end{aligned}
\tag{5b}
$$

$$
\begin{aligned}
F_{zx}(\xi_a,\eta_b) &= \gamma_0\xi_a\eta_b^0+\gamma_1\xi_a\eta_b^2+\cdots+\gamma_{n-1}\xi_a\eta_b^{2\cdot(n-1)}+\gamma_n\xi_a\eta_b^{2\cdot n}\\
&\quad+\gamma_{n+1}\xi_a^3\eta_b^0+\gamma_{n+2}\xi_a^3\eta_b^2+\cdots+\gamma_{2n-1}\xi_a^3\eta_b^{2\cdot(n-1)}\\
&\qquad\vdots\qquad\qquad\vdots\\
&\quad+\gamma_{l-n-1}\xi_a^{2\cdot n+1}\eta_b^0\\
&=\sum_{i=0}^{l-1}\gamma_i R_i(\xi_a,\eta_b)
\end{aligned}
\tag{5c}
$$

$$
\begin{aligned}
&l=\frac{(n+1)(n+2)}{2}\\
&G_0(\xi_a,\eta_b)=1\\
&G_1(\xi_a,\eta_b)=\eta_b^2,\ldots,G_{n+1}(\xi_a,\eta_b)=\xi_a^2,\ldots,\\
&G_{l-1}(\xi_a,\eta_b)=\xi_a^{2\cdot n}\\
&Q_0(\xi_a,\eta_b)=\eta_b\\
&Q_1(\xi_a,\eta_b)=\eta_b^3,\ldots,Q_{n+1}(\xi_a,\eta_b)=\xi_a^2\eta_b,\ldots,\\
&Q_{l-1}(\xi_a,\eta_b)=\xi_a^{2\cdot n}\eta_b\\
&R_0(\xi_a,\eta_b)=\xi_a\\
&R_1(\xi_a,\eta_b)=\xi_a\eta_b^2,\ldots,R_{n+1}(\xi_a,\eta_b)=\xi_a^3,\ldots,\\
&R_{l-1}(\xi_a,\eta_b)=\xi_a^{2\cdot n+1}
\end{aligned}
\tag{5d}
$$

Using the approximation method mentioned above, we obtain the following system of algebraic equations for the determination of unknown coefficients $\alpha_0\sim\alpha_i,\ \beta_0\sim\beta_i,\ \gamma_0\sim\gamma_i\ [i=1,2,\ldots,l,\ l=(1/2)(n+1)(n+2)]$, which can be determined by selecting a set of collocation points.

$$
\begin{aligned}
&\sum_{i=0}^{l}\left[\alpha_i\left(A_{zz,i}^{f_{zz}}+B_{zz,i}^{f_{zz}}\right)+\beta_iB_{zz,i}^{f_{yz}}+\gamma_iB_{zz,i}^{f_{zx}}\right]=-1\\
&\sum_{i=0}^{l}\left[\alpha_iB_{yz,i}^{f_{zz}}+\beta_i\left(A_{yz,i}^{f_{yz}}+B_{yz,i}^{f_{yz}}\right)+\gamma_i\left(A_{yz,i}^{f_{zx}}+B_{yz,i}^{f_{zx}}\right)\right]=0\\
&\sum_{i=0}^{l}\left[\alpha_iB_{zx,i}^{f_{zz}}+\beta_i\left(A_{zx,i}^{f_{yz}}+B_{zx,i}^{f_{yz}}\right)+\gamma_i\left(A_{zx,i}^{f_{zx}}+B_{zx,i}^{f_{zx}}\right)\right]=0
\end{aligned}
\tag{6a}
$$

The number of unknowns in Eq. (6a) are $3(l+1)$. As examples, $A_{zz,i}^{f_{zz}}$, $B_{zz,i}^{f_{zz}}$, $B_{zz,i}^{f_{yz}}$, $B_{zz,i}^{f_{zx}}$ are expressed as follows:

$$
\begin{aligned}
A_{zz,i}^{f_{zz}}&=\frac{b}{2\pi E(k)}\oint\!\!\oint_S\frac{G_i(\xi_a,\eta_b)}{r^3}\sqrt{1-\xi_a^2-\eta_b^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
B_{zz,i}^{f_{zz}}&=\frac{b}{8\pi E(k)}\iint_S K_{zz}^{f_{zz}}(\xi,\eta,x,y)G_i(\xi_a,\eta_b)\\
&\quad\times\sqrt{1-\xi_a^2-\eta_b^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
B_{zz,i}^{f_{yz}}&=\frac{bk^2}{8\pi C(k)}\iint_S K_{zz}^{f_{yz}}(\xi,\eta,x,y)Q_i(\xi_a,\eta_b)\\
&\quad\times\sqrt{1-\xi_a^2-\eta_b^2}\,\mathrm{d}\xi\,\mathrm{d}\eta\\
B_{zz,i}^{f_{zx}}&=\frac{bk^2}{8\pi B(k)}\iint_S K_{zx}^{f_{zx}}(\xi,\eta,x,y)R_i(\xi_a,\eta_b)\\
&\quad\times\sqrt{1-\xi_a^2-\eta_b^2}\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
\tag{6b}
$$

In Eq. (6b) the integral $B_{zz,i}^{f_{zz}}$ can be evaluated numerically because of no singularities in the integral. However $A_{zz,i}^{f_{zz}}$ cannot be evaluated by ordinary numerical procedure because they have hypersingularites of the form $r^{-3}$ when $x=\xi$ and $y=\eta$ (Hadamard, 1923). Therefore a similar method as shown in previous papers is applied (Noda and Miyoshi, 1996).

Fig. 2 shows boundary collocation points. The boundary conditions are considered at the intersection of the mesh on the $(x_a, y_b)$ plane in the region of $x_a^2+y_a^2<1$, $x_a^2\geqslant 0$, $y_a^2\geqslant 0$, where $x_a=x/a$, $y_b=y/b$. Fig. 2(a) shows $10\times 10$ mesh whose width is 0.1, and Fig. 2(b) shows $50\times 50$ mesh whose width is 0.02.

## 4. Numerical results and discussion

### 4.1. Dimensionless stress intensity factors

Numerical calculations have been carried out for changing $n$ in Eq. (5) when $a/b=1, 2, 4, 16, \infty$ and $\nu_1,\nu_2=0$–$0.5$. Numerical integrals have been performed using scientific subroutine library (FACOM SSL II DAQE etc.). In demonstrating the numerical results of stress intensity factor $K_{\mathrm{I}}$, $K_{\mathrm{II}}$, $K_{\mathrm{III}}$ the following dimensionless factor $F_{\mathrm{I}}$, $F_{\mathrm{II}}$, $F_{\mathrm{III}}$ will be used.

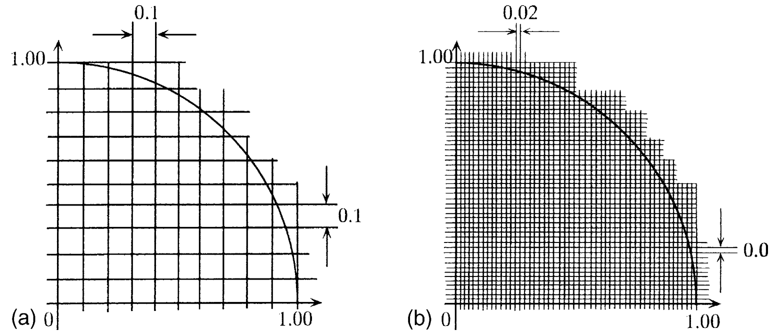



Fig. 2. Boundary collocation points. (a) 10 × 10 mesh (b) 50 × 50 mesh.

$$
\begin{aligned}
F_{\rm I}(\beta) &= \frac{K_{\rm I}(\beta)}{\sigma_z^\infty\sqrt{\pi b}} = \frac{F_{zz}}{E(k)}\left[\sin^2\beta + \left(\frac{b}{a}\right)^2\cos^2\beta\right]^{1/4} \\
F_{\rm II}(\beta) &= \frac{K_{\rm II}(\beta)}{\sigma_z^\infty\sqrt{\pi b}} \\
&= \left(F_{zx}\frac{k'\cos\beta}{B(k)} + F_{yz}\frac{\sin\beta}{C(k)}\right)\frac{k^2}{(1-k^2\cos^2\beta)^{1/4}} \\
F_{\rm III}(\beta) &= \frac{K_{\rm III}(\beta)|}{\sigma_z^\infty\sqrt{\pi b}} \\
&= \left(-F_{zx}\frac{\sin\beta}{B(k)} + F_{yz}\frac{k'\cos\beta}{C(k)}\right)\frac{(1-\nu)k^2}{(1-k^2\cos^2\beta)^{1/4}}
\end{aligned} \tag{7}
$$

In the following discussion, the maximum stress intensity factors $F_{\rm I}(\beta)$, $F_{\rm II}(\beta)$ appearing at $\beta = \pi/2$ will be mainly considered. In addition, the results using Murakami's $\sqrt{\text{area}}$ parameter will be also discussed (Murakami and Endo, 1983; Murakami and Nemat-Nasser, 1983; Murakami, 1985; Murakami and Isida, 1985; Murakami et al., 1988). Here, "area" is the area of crack.

$$
\begin{aligned}
F_{\rm I}^* &= \frac{K_{\rm I}}{\sigma_z^\infty\sqrt{\pi\sqrt{\text{area}}}} = \left(\frac{b}{\pi a}\right)^{1/4} \times F_{\rm I} \\
F_{\rm II}^* &= \frac{K_{\rm II}}{\sigma_z^\infty\sqrt{\pi\sqrt{\text{area}}}} = \left(\frac{b}{\pi a}\right)^{1/4} \times F_{\rm II} \\
F_{\rm III}^* &= \frac{K_{\rm III}}{\sigma_z^\infty\sqrt{\pi\sqrt{\text{area}}}} = \left(\frac{b}{\pi a}\right)^{1/4} \times F_{\rm III}
\end{aligned} \tag{8}
$$

### 4.2. Convergence and accuracy of the results

Tables 1 and 2 show convergence of stress intensity factors $F_{\rm I}(\beta)$, $F_{\rm II}(\beta)$ at $\beta = \pi/2$ when $a/b = 1$, $a/b = 16$, $\nu_1$, $\nu_2 = 0.3$ and $\mu_2/\mu_1 = 0$. Table 1(a) indicates that 10 × 10 boundary collocation points in Fig. 2(a) have convergence to the fourth digit when $h/2b \geqslant 0.3$. The convergence becomes worse as $h/2b \to 0$ and $a/b \to \infty$ due to the large effect of interface. On the other hand, Table 1(b) indicates that 50 × 50 boundary collocation points in Fig. 2(b) have convergence to the fourth digit when $a/b = 1$, and to the third digit when $a/b = 16$.

In the following calculation, the collocation points of 10 × 10 will be used when $b/a > 0.2$, and the ones of 50 × 50 will be used when $b/a \leqslant 0.2$.

In Table 3, the present results are compared with the solution of Sahin and Erdogan (1997) when $a/b = 1$, $\nu_1 = 0.3$, $\mu_2/\mu_1 = 0$. The results coincide with each other to the fourth digit when $h/2b \geqslant 0.1$. Fig. 3 indicates the compliance of the boundary conditions along the prospective crack surface. For $h/2b = 0.1$ the remaining stress $\sigma_z$ is less than $0.8 \times 10^{-2}$, and the remaining stresses $\tau_{yz}$, $\tau_{zx}$ are less than $0.6 \times 10^{-3}$ when $n = 7$. For $h/2b = 0.2$ $\sigma_z$ is less than $0.8 \times 10^{-4}$, and $\tau_{yz}$, $\tau_{zx}$ are less than $0.6 \times 10^{-5}$.

### 4.3. Effect of Poisson's ratio

Table 4 shows the results of different Poisson's ratio. The results vary depending on Poisson's ratio by about 11% when $a/b = 16$, $h/2b = 0.4$;

Table 1
Convergence of the results $F_I$, $F_{II}$ when $\mu_2/\mu_1 = 0$, $\beta = \pi/2$, $\nu_1$, $\nu_2 = 0.3$

| $a/b$ | | $n$ | $h/2b$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 | 2.0 |
| 1 | $F_I$ | 4 | 2.469 | 1.300 | 0.9867 | 0.8508 | 0.77816 | 0.66731 | 0.64142 |
| | | 5 | 2.473 | 1.299 | 0.9868 | 0.8508 | 0.77816 | 0.66731 | 0.64142 |
| | | 6 | 2.495 | 1.299 | 0.9868 | 0.8508 | 0.77816 | 0.66731 | 0.64142 |
| | $F_{II}$ | 4 | 1.087 | 0.346 | 0.1613 | 0.08787 | 0.05196 | 0.00696 | 0.00058 |
| | | 5 | 1.099 | 0.346 | 0.1613 | 0.08787 | 0.05196 | 0.00696 | 0.00058 |
| | | 6 | 1.110 | 0.346 | 0.1613 | 0.08787 | 0.05196 | 0.00696 | 0.00058 |
| 16 | $F_I$ | 4 | 5.96 | 2.902 | 2.0748 | 1.7085 | 1.5062 | 1.1590 | 1.04000 |
| | | 5 | 5.98 | 2.897 | 2.0757 | 1.7090 | 1.5064 | 1.1589 | 1.04000 |
| | | 6 | 6.03 | 2.896 | 2.0748 | 1.7087 | 1.5063 | 1.1589 | 1.04000 |
| | $F_{II}$ | 4 | 2.98 | 0.993 | 0.4912 | 0.2874 | 0.1838 | 0.03624 | 0.00521 |
| | | 5 | 3.01 | 0.990 | 0.4917 | 0.2876 | 0.1837 | 0.03623 | 0.00521 |
| | | 6 | 2.99 | 0.992 | 0.4913 | 0.2874 | 0.1838 | 0.03623 | 0.00521 |

Number of collocation points $10 \times 10$.

Table 2
Convergence of the results $F_I$, $F_{II}$ when $\mu_2/\mu_1 = 0$, $\beta = \pi/2$, $\nu_1$, $\nu_2 = 0.3$

| $a/b$ | | $n$ | $h/2b$ | |
|---|---|---|---|---|
| | | | 0.1 | 0.2 |
| 1 | $F_I$ | 4 | 2.463 | 1.299 |
| | | 5 | 2.463 | 1.299 |
| | | 6 | 2.461 | 1.299 |
| | $F_{II}$ | 4 | 1.105 | 0.3457 |
| | | 5 | 1.106 | 0.3457 |
| | | 6 | 1.105 | 0.3457 |
| 16 | $F_I$ | 4 | 5.956 | 2.898 |
| | | 5 | 5.932 | 2.892 |
| | | 6 | 5.943 | 2.898 |
| | $F_{II}$ | 4 | 3.025 | 0.9902 |
| | | 5 | 3.016 | 0.9904 |
| | | 6 | 3.023 | 0.9903 |

Number of collocation points $50 \times 50$.

however, the results vary about 5% when $a/b = 1$, $h/2b = 0.4$. The effect is not very large even when Poisson's ratios are changed extremely from $(\nu_1, \nu_2) = (0, 0.5)$ to $(\nu_1, \nu_2) = (0.5, 0)$. Therefore in the following calculations we simply assume $\nu_1$, $\nu_2 = 0.3$. Fig. 4 shows examples of the effect of Poisson's ratio. When $a/b = 1$, $h/2b = 0.4$, $\mu_2/\mu_1 = 0$, the results vary only about 0.1% and increase with increasing $\nu_1$. On the other hand,

Table 3
Results of a penny-shaped crack in a semi-infinite body

| $h/2b$ | $F_I$ | | $F_{II}$ | |
|---|---|---|---|---|
| | Sahin–Erdogan | Present analysis | Sahin–Erdogan | Present analysis |
| 5 | 0.6369 | 0.6369 | 0.0000 | 0.0000 |
| 2 | | 0.6414 | | 0.0006 |
| 1 | 0.6673 | 0.6673 | 0.0070 | 0.0070 |
| 0.5 | 0.7781 | 0.7782 | 0.0520 | 0.0520 |
| 0.4 | | 0.8507 | | 0.8787 |
| 0.375 | 0.8763 | 0.8766 | 0.1013 | 0.1013 |
| 0.3 | | 0.9868 | | 0.1613 |
| 0.25 | 1.1061 | 1.1061 | 0.2297 | 0.2297 |
| 0.2 | | 1.2991 | | 0.3457 |
| 0.125 | 1.9620 | 1.9611 | 0.7704 | 0.7700 |
| 0.1 | | 2.461 | | 1.105 |
| 0.05 | 5.5317 | 5.50 | 3.2759 | 3.24 |

when $a/b = 16$, $h/2b = 0.4$, $\mu_2/\mu_1 = \infty$, $F_I$ varies by about 7% and becomes largest at $\nu_1 = 0.18$.

### 4.4. Stress intensity factor of an elliptical crack parallel to a bimaterial interface

Table 5 (Panels a–c) shows the maximum stress intensity factors $F_I$, $F_{II}$, $F_I^*$, $F_{II}^*$ at $\beta = \pi/2$ when $a/b = 1$, 2, 4, 16, $\infty$, $\mu_2/\mu_1 = 0$, 0.5, 2, $\infty$, and $h/2b = 0.1$–$\infty$. Also, the maximum $F_{III}$, $F_{III}^*$ values are indicated with their position in the range

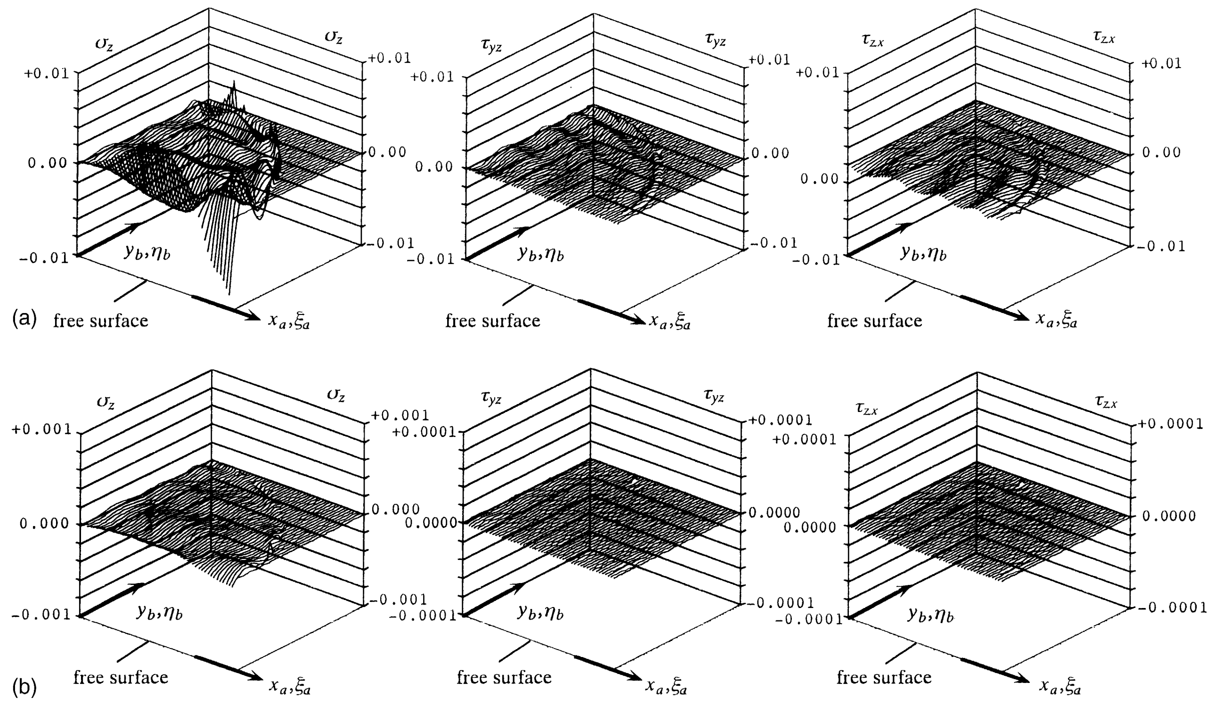



Fig. 3. Compliance of boundary condition $\sigma_z^\infty \cong 0$, $\tau_{yz}^\infty \cong 0$, $\tau_{zx}^\infty \cong 0$ in Fig. 1 when $n = 7$, $a/b = 1$, $\nu_1$, $\nu_2 = 0.3$. (a) $\sigma_z^\infty \cong 0$, $\tau_{yz}^\infty \cong 0$, $\tau_{zx}^\infty \cong 0$ when $h/2b = 0.1$. (b) $\sigma_z^\infty \cong 0$, $\tau_{yz}^\infty \cong 0$, $\tau_{zx}^\infty \cong 0$ when $h/2b = 0.2$.

Table 4
Dimensionless stress intensity factors $F_{\mathrm{I}}$, $F_{\mathrm{II}}$ in Fig. 1

| | | $a/b = 16$ | | | | $a/b = 1$ | | |
|---|---|---|---|---|---|---|---|---|
| | | $h/2b = 0.4$ | | | | $h/2b = 0.1$, $\mu_2/\mu_1 = 0.5$ | $h/2b = 0.4$, $\mu_2/\mu_1 = 0.5$ | $h/2b = 1.0$, $\mu_2/\mu_1 = 0.5$ |
| | | $\mu_2/\mu_1 = 0$ | $\mu_2/\mu_1 = 0.5$ | $\mu_2/\mu_1 = 2.0$ | $\mu_2/\mu_1 = \infty$ | | | |
| $F_{\mathrm{I}}$ | $\nu_1 = 0.0$, $\nu_2 = 0.0$ | 1.7090 | 1.0857 | 0.9251 | 0.798 | 0.7243 | 0.6710 | 0.6429 |
| | $\nu_1 = 0.5$, $\nu_2 = 0.5$ | 1.7092 | 1.1316 | 0.8938 | 0.760 | 0.7563 | 0.6901 | 0.6465 |
| | $\nu_1 = 0.0$, $\nu_2 = 0.5$ | 1.7090 | 1.0352 | 0.8794 | 0.798 | 0.6544 | 0.6586 | 0.6415 |
| | $\nu_1 = 0.5$, $\nu_2 = 0.0$ | 1.7092 | 1.1628 | 0.9168 | 0.760 | 0.8093 | 0.6971 | 0.6472 |
| | $\nu_1 = 0.3$, $\nu_2 = 0.3$ | 1.7090 | 1.1073 | 0.9134 | 0.800 | 0.7397 | 0.6800 | 0.6446 |
| $F_{\mathrm{II}}$ | $\nu_1 = 0.0$, $\nu_2 = 0.0$ | 0.288 | 0.0371 | −0.0280 | −0.084 | 0.0513 | 0.0141 | 0.0014 |
| | $\nu_1 = 0.5$, $\nu_2 = 0.5$ | 0.287 | 0.0530 | −0.0362 | −0.082 | 0.0507 | 0.0214 | 0.0022 |
| | $\nu_1 = 0.0$, $\nu_2 = 0.5$ | 0.288 | 0.0119 | −0.0509 | −0.084 | −0.0201 | 0.0104 | 0.0011 |
| | $\nu_1 = 0.5$, $\nu_2 = 0.0$ | 0.287 | 0.0683 | −0.0247 | −0.082 | 0.1074 | 0.0249 | 0.0024 |
| | $\nu_1 = 0.3$, $\nu_2 = 0.3$ | 0.287 | 0.0446 | −0.0308 | −0.075 | 0.0520 | 0.0176 | 0.0018 |

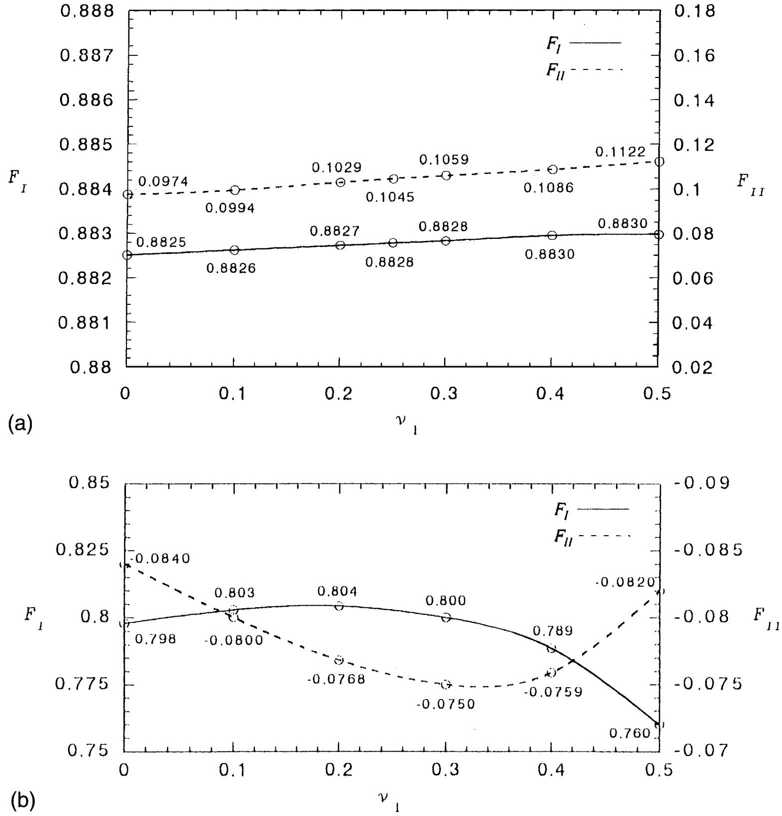



Fig. 4. (a) Effect of Poisson's ratio when $a/b = 1$, $h/2b = 0.4$, $\mu_2/\mu_1 = 0$. (b) Effect of Poisson's ratio when $a/b = 16$, $h/2b = 0.4$, $\mu_2/\mu_1 = \infty$.

$\beta = \pi/20$–$\pi/4$. The results of $a/b = \infty$ is obtained from a two-dimensional program used in the previous study (Oda et al., 1998). If $h/2b \leqslant 0.5$, $\mu_2/\mu_1 \leqslant 0.1$, the $F_{\mathrm{II}}$ value is larger than 10% of the $F_{\mathrm{I}}$ value, and cannot be ignored. In other cases, however, the value of $F_{\mathrm{II}}$ is only several percent or less of the value of $F_{\mathrm{I}}$. The $F_{\mathrm{III}}$ values are less than the values of $F_{\mathrm{II}}$. In Table 5 (Panel c), the largest value of $F_{\mathrm{III}} = 0.1547$ when $\mu_2/\mu_1 = 0$, $h/2b = 0.1$, $a/b = 2$.

In Table 5 (Panels a–c), the ratios of the results of $a/b = 1$ and $a/b = \infty$ are also shown as $(a/b = 1)/(a/b = \infty)$. The ratio of $F_{\mathrm{I}}$ is 0.41–0.69. On the other hand, the ratio of $F_{\mathrm{I}}^*$ is $0.97$–$1.10 \cong 1$ unless $h/2b \leqslant 1.0$, $\mu_2/\mu_1 \leqslant 0.1$. Fig. 5 shows $F_{\mathrm{I}}$, $F_{\mathrm{II}}$ vs. $h/2b$, and Fig. 6 shows $F_{\mathrm{I}}^*$, $F_{\mathrm{II}}^*$ vs. $h/2b$ when $\mu_2/\mu_1 = 0, \infty$. It is seen $F_{\mathrm{I}}^*$, $F_{\mathrm{II}}^*$ is insensitive to $a/b$. The $\sqrt{\mathrm{area}}$ parameter $F_{\mathrm{I}}^*$ is found to be effective for engineering use because the effect of $a/b$ on $F_{\mathrm{I}}^*$ is small. In other words, different shaped cracks have almost the same values of $F_{\mathrm{I}}^*$.

Figs. 7–9 show the distribution of the stress intensity factors $F_{\mathrm{I}}$, $F_{\mathrm{II}}$, $F_{\mathrm{III}}$ when $h/2b = 0.1$, 0.5, $\infty$. The maximum values of $F_{\mathrm{I}}$, $F_{\mathrm{II}}$ appearing at $\beta = \pi/2$ becomes greatly influenced by the interface according to $h/2b \to 0$ especially for large value of $a/b$.

## 5. Conclusion

In this study an elliptical crack parallel to a bimaterial interface was considered. The stress intensity factors were calculated systematically with varying the aspect ratio of crack, elastic constants of materials, and the distance between the crack and interface. The conclusion can be made as follows.

Table 5
Dimensionless stress intensity factors when $\nu_1$, $\nu_2 = 0.3$ in Fig. 1. Panel a: $F_{\mathrm{I}}$, $F_{\mathrm{I}}^*$ at $\beta = \pi/2$. Panel b: $F_{\mathrm{II}}$, $F_{\mathrm{II}}^*$ at $\beta = \pi/2$. Panel c: $F_{\mathrm{III}}$, $F_{\mathrm{III}}^*$ at $\beta = \pi/20$–$\pi/4$

| $a/b$ | $F_{\mathrm{I}}$ | | | | | | | | $F_{\mathrm{I}}^*$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mu_2/\mu_1 = 0$ | $\mu_2/\mu_1 = 0.01$ | $\mu_2/\mu_1 = 0.05$ | $\mu_2/\mu_1 = 0.1$ | $\mu_2/\mu_1 = 0.3$ | $\mu_2/\mu_1 = 0.5$ | $\mu_2/\mu_1 = 1.0$ | $\mu_2/\mu_1 = 2.0$ | $\mu_2/\mu_1 = 0$ | $\mu_2/\mu_1 = 0.01$ | $\mu_2/\mu_1 = 0.05$ | $\mu_2/\mu_1 = 0.1$ | $\mu_2/\mu_1 = 0.3$ | $\mu_2/\mu_1 = 0.5$ | $\mu_2/\mu_1 = 1.0$ | $\mu_2/\mu_1 = 2.0$ |
| *Panel a* | | | | | | | | | | | | | | | | |
| $h/2b = 0.1$ | | | | | | | | | | | | | | | | |
| 1 | 2.461 | 2.067 | 1.447 | 1.175 | 0.8457 | 0.7397 | 0.5699 | 0.4779 | 1.8485 | 1.5526 | 1.0869 | 0.8826 | 0.6352 | 0.5556 | 0.4281 | 0.3590 |
| 2 | 4.830 | | | | | 0.9756 | 0.7295 | 0.5975 | 3.0507 | | | | | 0.6132 | 0.4608 | 0.3774 |
| 4 | 5.692 | | | | | 1.1041 | 0.8221 | 0.6694 | 3.0232 | | | | | 0.5864 | 0.4366 | 0.3555 |
| 16 | 5.94 | | | | | 1.175 | 0.875 | 0.7122 | 2.8103 | | | | | 0.5556 | 0.4138 | 0.3368 |
| $\to \infty$ | 5.95 | 4.2893 | 2.5728 | 1.9972 | 1.3733 | 1.183 | 0.882 | 0.7175 | 2.8136 | 2.0283 | 1.2166 | 0.9444 | 0.6494 | 0.5594 | 0.4171 | 0.3390 |
| $(a/b = 1)/(a/b = \infty)$ | 0.4136 | 0.4819 | 0.5624 | 0.5878 | 0.6157 | 0.6253 | 0.6461 | 0.6665 | 0.6570 | 0.7655 | 0.8934 | 0.9346 | 0.9781 | 0.9932 | 1.0263 | 1.0590 |
| $h/2b = 0.2$ | | | | | | | | | | | | | | | | |
| 1 | 1.2991 | 1.2477 | 1.1001 | 0.9875 | 0.7926 | 0.7160 | 0.5837 | 0.5105 | 0.9758 | 0.9372 | 0.8263 | 0.7417 | 0.5953 | 0.5378 | 0.4384 | 0.3834 |
| 2 | 2.3369 | | | | | 0.9554 | 0.7424 | 0.6299 | 1.4760 | | | | | 0.6034 | 0.4689 | 0.3979 |
| 4 | 2.7735 | | | | | 1.0829 | 0.8355 | 0.7033 | 1.4731 | | | | | 0.5752 | 0.4438 | 0.3735 |
| 16 | 2.898 | | | | | 1.152 | 0.890 | 0.749 | 1.3704 | | | | | 0.5447 | 0.4209 | 0.3542 |
| $\to \infty$ | 2.9052 | 2.6764 | 2.1307 | 1.7958 | 1.3216 | 1.1594 | 0.8966 | 0.7546 | 1.3738 | 1.2656 | 1.0075 | 0.8492 | 0.6249 | 0.5482 | 0.4240 | 0.3568 |
| $(a/b = 1)/(a/b = \infty)$ | 0.4472 | 0.4662 | 0.5163 | 0.5499 | 0.5997 | 0.6176 | 0.6510 | 0.6765 | 0.7296 | 0.7405 | 0.8201 | 0.8734 | 0.9526 | 0.9810 | 1.0340 | 1.0746 |
| $h/2b = 0.3$ | | | | | | | | | | | | | | | | |
| 1 | 0.9868 | 0.9695 | 0.9125 | 0.8604 | 0.7483 | 0.6958 | 0.5945 | 0.5355 | 0.7412 | 0.7282 | 0.6854 | 0.6463 | 0.5621 | 0.5226 | 0.4465 | 0.4022 |
| 2 | 1.6567 | | | | | 0.9347 | 0.7535 | 0.6551 | 1.0464 | | | | | 0.5904 | 0.4759 | 0.4138 |
| 4 | 1.9806 | | | | | 1.0321 | 0.8468 | 0.729 | 1.0519 | | | | | 0.5641 | 0.4498 | 0.3872 |
| 16 | 2.075 | | | | | 1.129 | 0.9023 | 0.776 | 0.9812 | | | | | 0.5339 | 0.4267 | 0.3669 |
| $\to \infty$ | 2.0809 | 2.0046 | 1.7766 | 1.5950 | 1.2677 | 1.1365 | 0.9090 | 0.7828 | 0.9840 | 0.9479 | 0.8401 | 0.7542 | 0.5995 | 0.5374 | 0.4298 | 0.3702 |
| $(a/b = 1)/(a/b = \infty)$ | 0.4742 | 0.4835 | 0.5136 | 0.5394 | 0.5903 | 0.6122 | 0.6540 | 0.6841 | 0.7533 | 0.7682 | 0.8159 | 0.8569 | 0.9376 | 0.9725 | 1.0389 | 1.0864 |
| $h/2b = 0.4$ | | | | | | | | | | | | | | | | |
| 1 | 0.8507 | 0.8424 | 0.8134 | 0.7848 | 0.7159 | 0.6800 | 0.6043 | 0.5565 | 0.6390 | 0.6327 | 0.6110 | 0.5895 | 0.5377 | 0.5108 | 0.4539 | 0.4180 |
| 2 | 1.3540 | | | | | 0.9143 | 0.7641 | 0.6777 | 0.8552 | | | | | 0.5775 | 0.4826 | 0.4280 |
| 4 | 1.6254 | | | | | 1.0417 | 0.8573 | 0.7520 | 0.8633 | | | | | 0.5533 | 0.4553 | 0.3994 |
| 16 | 1.7090 | | | | | 1.1073 | 0.9134 | 0.800 | 0.8129 | | | | | 0.5236 | 0.4319 | 0.3783 |
| $\to \infty$ | 1.7138 | 1.6764 | 1.5547 | 1.4457 | 1.2180 | 1.1145 | 0.9204 | 0.8073 | 0.8104 | 0.7927 | 0.7352 | 0.6836 | 0.5760 | 0.5270 | 0.4352 | 0.3817 |
| $(a/b = 1)/(a/b = \infty)$ | 0.4964 | 0.5025 | 0.5231 | 0.5429 | 0.5878 | 0.5939 | 0.6566 | 0.6893 | 0.7885 | 0.7982 | 0.8311 | 0.8623 | 0.9335 | 0.9693 | 1.0429 | 1.0951 |
| $h/2b = 0.5$ | | | | | | | | | | | | | | | | |
| 1 | 0.7881 | 0.7733 | 0.7561 | 0.7385 | 0.6935 | 0.6684 | 0.6120 | 0.5741 | 0.5920 | 0.5808 | 0.5679 | 0.5547 | 0.5209 | 0.5021 | 0.4597 | 0.4312 |
| 2 | 1.1879 | | | | | 0.8963 | 0.7742 | 0.6995 | 0.7503 | | | | | 0.5661 | 0.4890 | 0.4418 |
| 4 | 1.4278 | | | | | 1.0234 | 0.8674 | 0.7734 | 0.7583 | | | | | 0.5436 | 0.4607 | 0.4108 |
| 16 | 1.5063 | | | | | 1.0881 | 0.9240 | 0.8233 | 0.7123 | | | | | 0.5145 | 0.4369 | 0.3893 |
| $\to \infty$ | 1.5110 | 1.4888 | 1.4132 | 1.3412 | 1.1766 | 1.0952 | 0.9311 | 0.8302 | 0.7145 | 0.7040 | 0.6683 | 0.6342 | 0.5564 | 0.5179 | 0.4403 | 0.3926 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(a/b=1)/(a/b=\infty)$ | 0.5216 | 0.5194 | 0.5414 | 0.5506 | 0.5894 | 0.6103 | 0.6573 | 0.6915 | 0.8286 | 0.8250 | 0.8498 | 0.8746 | 0.9362 | 0.9695 | 1.0441 | 1.0983 |
| $h/2b=1.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.66731 | 0.66644 | 0.66328 | 0.65990 | 0.65044 | 0.64461 | 0.62992 | 0.61883 | 0.50123 | 0.50058 | 0.49821 | 0.49567 | 0.48856 | 0.48418 | 0.47315 | 0.46482 |
| 2 | 0.91536 | | | | | 0.84817 | 0.80735 | 0.77768 | 0.57816 | | | | | 0.53572 | 0.50994 | 0.49067 |
| 4 | 1.08299 | | | | | 0.96904 | 0.90318 | 0.85624 | 0.57520 | | | | | 0.51468 | 0.47902 | 0.45477 |
| 16 | 1.1589 | | | | | 1.03320 | 0.96027 | 0.90720 | 0.54801 | | | | | 0.48857 | 0.45408 | 0.42899 |
| $\to\infty$ | 1.16332 | 1.15826 | 1.14002 | 1.12093 | 1.06976 | 1.03975 | 0.96772 | 0.91518 | 0.55010 | 0.54771 | 0.53908 | 0.53006 | 0.50586 | 0.49167 | 0.45761 | 0.43276 |
| $(a/b=1)/(a/b=\infty)$ | 0.57363 | 0.57535 | 0.58183 | 0.58871 | 0.60799 | 0.61997 | 0.65093 | 0.67618 | 0.91162 | 0.91395 | 0.92419 | 0.93512 | 0.96580 | 0.98477 | 1.0339 | 1.07408 |
| $h/2b=2.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.64142 | 0.64130 | 0.64082 | 0.64030 | 0.63883 | 0.63790 | 0.63552 | 0.63367 | 0.48179 | 0.48170 | 0.48134 | 0.48095 | 0.47984 | 0.47914 | 0.47736 | 0.47597 |
| 2 | 0.84108 | | | | | 0.82982 | 0.82225 | 0.81643 | 0.53124 | | | | | 0.52413 | 0.51935 | 0.51567 |
| 4 | 0.96518 | | | | | 0.94118 | 0.92529 | 0.91311 | 0.51263 | | | | | 0.49989 | 0.49145 | 0.48498 |
| 16 | 1.04000 | | | | | 1.00538 | 0.98233 | 0.96418 | 0.49179 | | | | | 0.47541 | 0.46452 | 0.45593 |
| $\to\infty$ | 1.04507 | 1.04383 | 1.03927 | 1.03437 | 1.02055 | 1.01196 | 0.98978 | 0.97201 | 0.49418 | 0.49360 | 0.49144 | 0.48912 | 0.48259 | 0.47853 | 0.46804 | 0.45964 |
| $(a/b=1)/(a/b=\infty)$ | 0.61376 | 0.61437 | 0.61659 | 0.61902 | 0.62594 | 0.63036 | 0.64208 | 0.65192 | 0.97493 | 0.97589 | 0.97945 | 0.98330 | 0.99430 | 1.00127 | 1.01991 | 1.03552 |
| $h/2b=\infty$ | | | | | | | | | | | | | | | | |
| 1 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.47818 | 0.47818 | 0.47818 | 0.47818 | 0.47818 | 0.47818 | 0.47818 | 0.47818 |
| 2 | 0.82572 | 0.82572 | 0.82572 | 0.82572 | 0.82572 | 0.82572 | 0.82572 | 0.82572 | 0.52154 | 0.52154 | 0.52154 | 0.52154 | 0.52154 | 0.52154 | 0.52154 | 0.52154 |
| 4 | 0.93297 | 0.93297 | 0.93297 | 0.93297 | 0.93297 | 0.93297 | 0.93297 | 0.93297 | 0.49552 | 0.49552 | 0.49552 | 0.49552 | 0.49552 | 0.49552 | 0.49552 | 0.49552 |
| 16 | 0.99275 | 0.99275 | 0.99275 | 0.99275 | 0.99275 | 0.99275 | 0.99275 | 0.99275 | 0.46944 | 0.46944 | 0.46944 | 0.46944 | 0.46944 | 0.46944 | 0.46944 | 0.46944 |
| $\to\infty$ | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 0.47287 | 0.47287 | 0.47287 | 0.47287 | 0.47287 | 0.47287 | 0.47287 | 0.47287 |
| $(a/b=1)/(a/b=\infty)$ | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 0.63662 | 1.01123 | 1.01123 | 1.01123 | 1.01123 | 1.01123 | 1.01123 | 1.01123 | 1.01123 |
| *Panel b* | | | | | | | | | | | | | | | | |
| | $F_{II}$ | | | | | | | | | $F^*_{II}$ | | | | | | |
| $h/2b=0.1$ | | | | | | | | | | | | | | | | |
| 1 | 1.105 | 0.8462 | 0.4509 | 0.2877 | 0.1063 | 0.0520 | −0.0344 | −0.0847 | 0.8300 | 0.6356 | 0.3387 | 0.2161 | 0.0798 | 0.0391 | −0.0258 | −0.0636 |
| 2 | 2.378 | | | | | 0.0735 | −0.0485 | −0.1190 | 1.5020 | | | | | 0.0464 | −0.0306 | −0.0752 |
| 4 | 2.874 | | | | | 0.0855 | −0.0569 | −0.1397 | 1.5265 | | | | | 0.0454 | −0.0302 | −0.0742 |
| 16 | 3.02 | | | | | 0.092 | −0.0616 | −0.152 | 1.4281 | | | | | 0.0435 | −0.0291 | −0.0719 |
| $\to\infty$ | 3.303 | 1.9460 | 0.8634 | 0.5259 | 0.1897 | 0.093 | −0.0622 | −0.153 | 1.5619 | 0.9202 | 0.4083 | 0.2487 | 0.0897 | 0.0440 | −0.0293 | −0.0723 |
| $(a/b=1)/(a/b=\infty)$ | 0.3345 | 0.4348 | 0.5222 | 0.5471 | 0.5604 | 0.5591 | 0.5548 | 0.5536 | 0.5314 | 0.6907 | 0.8295 | 0.8689 | 0.8896 | 0.8886 | 0.8805 | 0.8797 |
| $h/2b=0.2$ | | | | | | | | | | | | | | | | |
| 1 | 0.3457 | 0.3171 | 0.2362 | 0.1758 | 0.0754 | 0.0378 | −0.0250 | −0.0607 | 0.2597 | 0.2382 | 0.1774 | 0.1320 | 0.0566 | 0.0284 | −0.0188 | −0.0456 |
| 2 | 0.7456 | | | | | 0.0577 | −0.0367 | −0.0882 | 0.4709 | | | | | 0.0364 | −0.0232 | −0.0557 |

Table 5 (*continued*)

| $a/b$ | $F_{II}$ | | | | | | | | $F_{II}^*$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mu_2/\mu_1 = 0$ | $\mu_2/\mu_1 = 0.01$ | $\mu_2/\mu_1 = 0.05$ | $\mu_2/\mu_1 = 0.1$ | $\mu_2/\mu_1 = 0.3$ | $\mu_2/\mu_1 = 0.5$ | $\mu_2/\mu_1 = 1.0$ | $\mu_2/\mu_1 = 2.0$ | $\mu_2/\mu_1 = 0$ | $\mu_2/\mu_1 = 0.01$ | $\mu_2/\mu_1 = 0.05$ | $\mu_2/\mu_1 = 0.1$ | $\mu_2/\mu_1 = 0.3$ | $\mu_2/\mu_1 = 0.5$ | $\mu_2/\mu_1 = 1.0$ | $\mu_2/\mu_1 = 2.0$ |
| 4 | 0.9327 | | | | | 0.0678 | −0.0434 | −0.1048 | 0.4954 | | | | | 0.0360 | −0.0230 | −0.0557 |
| 16 | 0.9903 | | | | | 0.0731 | −0.0473 | −0.1150 | 0.4683 | | | | | 0.0346 | −0.0224 | −0.0544 |
| $\to\infty$ | 0.9940 | 0.8671 | 0.5688 | 0.3908 | 0.1509 | 0.0737 | −0.0477 | −0.1161 | 0.4700 | 0.4100 | 0.2690 | 0.1848 | 0.0714 | 0.0349 | −0.0226 | −0.0549 |
| $(a/b=1)/(a/b=\infty)$ | 0.3478 | 0.3657 | 0.4153 | 0.4498 | 0.4997 | 0.5129 | 0.5241 | 0.5228 | 0.5526 | 0.5810 | 0.6595 | 0.7143 | 0.7927 | 0.8137 | 0.8319 | 0.8306 |
| $h/2b = 0.3$ | | | | | | | | | | | | | | | | |
| 1 | 0.1613 | 0.1531 | 0.1261 | 0.1016 | 0.0499 | 0.0263 | −0.0184 | −0.0450 | 0.1212 | 0.1150 | 0.0947 | 0.0763 | 0.0375 | 0.0198 | −0.0138 | −0.0338 |
| 2 | 0.3512 | | | | | 0.0442 | −0.0290 | −0.0693 | 0.2218 | | | | | 0.0279 | −0.0183 | −0.0438 |
| 4 | 0.4570 | | | | | 0.0535 | −0.0350 | −0.0838 | 0.2427 | | | | | 0.0284 | −0.0186 | −0.0445 |
| 16 | 0.491 | | | | | 0.058 | −0.038 | −0.092 | 0.2322 | | | | | 0.0274 | −0.0180 | −0.0435 |
| $\to\infty$ | 0.4936 | 0.4572 | 0.3495 | 0.2648 | 0.1162 | 0.0586 | −0.0387 | −0.0937 | 0.2334 | 0.2162 | 0.1653 | 0.1252 | 0.0549 | 0.0277 | −0.0183 | −0.0443 |
| $(a/b=1)/(a/b=\infty)$ | 0.3268 | 0.3349 | 0.3608 | 0.3837 | 0.4294 | 0.4488 | 0.4755 | 0.4803 | 0.5193 | 0.5319 | 0.5729 | 0.6094 | 0.6831 | 0.7148 | 0.7541 | 0.7630 |
| $h/2b = 0.4$ | | | | | | | | | | | | | | | | |
| 1 | 0.0879 | 0.0844 | 0.0723 | 0.0605 | 0.0322 | 0.0176 | −0.0130 | −0.0326 | 0.0660 | 0.0634 | 0.0543 | 0.0454 | 0.0242 | 0.0132 | −0.0098 | −0.0245 |
| 2 | 0.1941 | | | | | 0.0322 | −0.0223 | −0.0540 | 0.1226 | | | | | 0.0210 | −0.0141 | −0.0341 |
| 4 | 0.2631 | | | | | 0.0406 | −0.0278 | −0.0672 | 0.1397 | | | | | 0.0216 | −0.0148 | −0.0357 |
| 16 | 0.287 | | | | | 0.0446 | −0.0308 | −0.075 | 0.1357 | | | | | 0.0211 | −0.0146 | −0.0355 |
| $\to\infty$ | 0.2890 | 0.2735 | 0.2231 | 0.1783 | 0.0861 | 0.0450 | −0.0311 | −0.0761 | 0.1367 | 0.1293 | 0.1055 | 0.0843 | 0.0407 | 0.0213 | −0.0147 | −0.0360 |
| $(a/b=1)/(a/b=\infty)$ | 0.3042 | 0.3086 | 0.3241 | 0.3393 | 0.3740 | 0.3911 | 0.4180 | 0.4284 | 0.4828 | 0.4903 | 0.5147 | 0.5386 | 0.5946 | 0.6197 | 0.6667 | 0.6806 |
| $h/2b = 0.5$ | | | | | | | | | | | | | | | | |
| 1 | 0.0520 | 0.0502 | 0.0438 | 0.0373 | 0.0208 | 0.0116 | −0.0090 | −0.0230 | 0.0391 | 0.0377 | 0.0329 | 0.0280 | 0.0156 | 0.0087 | −0.0068 | −0.0173 |
| 2 | 0.1167 | | | | | 0.0227 | −0.0166 | −0.0410 | 0.0737 | | | | | 0.0143 | −0.0105 | −0.0260 |
| 4 | 0.1649 | | | | | 0.0301 | −0.0216 | −0.0530 | 0.0876 | | | | | 0.0160 | −0.0115 | −0.0281 |
| 16 | 0.1838 | | | | | 0.0336 | −0.0243 | −0.0602 | 0.0869 | | | | | 0.0159 | −0.0115 | −0.0285 |
| $\to\infty$ | 0.1849 | 0.1767 | 0.1490 | 0.1228 | 0.0632 | 0.0339 | −0.0246 | −0.0610 | 0.0874 | 0.0836 | 0.0705 | 0.0581 | 0.0299 | 0.0160 | −0.0116 | −0.0288 |
| $(a/b=1)/(a/b=\infty)$ | 0.2812 | 0.2841 | 0.2940 | 0.3037 | 0.3291 | 0.3422 | 0.3658 | 0.3770 | 0.4474 | 0.4510 | 0.4667 | 0.4819 | 0.5217 | 0.5438 | 0.5862 | 0.6007 |
| $h/2b = 1.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.00696 | 0.00677 | 0.00605 | 0.00529 | 0.00314 | 0.00182 | 0.00152 | −0.00406 | 0.00523 | 0.00509 | 0.00454 | 0.00397 | 0.00236 | 0.00137 | −0.00114 | −0.00305 |
| 2 | 0.01653 | | | | | 0.00414 | 0.00340 | −0.00889 | 0.01044 | | | | | 0.00261 | −0.00215 | −0.00562 |
| 4 | 0.02801 | | | | | 0.00682 | 0.00550 | −0.01426 | 0.01488 | | | | | 0.00362 | −0.00292 | −0.00757 |
| 16 | 0.03623 | | | | | 0.00878 | 0.00713 | −0.01871 | 0.01713 | | | | | 0.00415 | −0.00337 | −0.00885 |
| $\to\infty$ | 0.03675 | 0.03561 | 0.03151 | 0.02721 | 0.01570 | 0.00895 | 0.00728 | −0.01921 | 0.01738 | 0.01684 | 0.01490 | 0.01287 | 0.00742 | 0.00423 | −0.00344 | −0.00908 |
| $(a/b=1)/(a/b=\infty)$ | 0.18939 | 0.19012 | 0.19200 | 0.19441 | 0.20000 | 0.20335 | 0.20879 | 0.21135 | 0.30092 | 0.30226 | 0.30470 | 0.30847 | 0.31806 | 0.32388 | 0.33139 | 0.33590 |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h/2b = 2.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.00059 | 0.00057 | 0.00051 | 0.00045 | 0.00027 | 0.00016 | 0.00013 | −0.00036 | 0.00044 | 0.00043 | 0.00038 | 0.00034 | 0.00020 | 0.00012 | −0.00010 | −0.00027 |
| 2 | 0.00142 | | | | | 0.00038 | 0.00032 | −0.00086 | 0.00090 | | | | | 0.00024 | −0.00020 | −0.00054 |
| 4 | 0.00276 | | | | | 0.00073 | 0.00062 | −0.00164 | 0.00147 | | | | | 0.00039 | −0.00033 | −0.00087 |
| 16 | 0.00521 | | | | | 0.00138 | 0.00117 | −0.00314 | 0.00246 | | | | | 0.00065 | −0.00055 | −0.00148 |
| $\to\infty$ | 0.00548 | 0.00532 | 0.00477 | 0.00418 | 0.00250 | 0.00145 | 0.00124 | −0.00339 | 0.00259 | 0.00252 | 0.00226 | 0.00198 | 0.00118 | 0.00069 | −0.00059 | −0.00160 |
| $(a/b = 1)/$ $(a/b = \infty)$ | 0.10766 | 0.10714 | 0.10692 | 0.10766 | 0.10800 | 0.11034 | 0.10484 | 0.10619 | 0.16988 | 0.17063 | 0.16814 | 0.17172 | 0.16949 | 0.17391 | 0.16949 | 0.16875 |
| $h/2b = \infty$ | | | | | | | | | | | | | | | | |
| 1 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 2 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 4 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 16 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $\to\infty$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $(a/b = 1)/$ $(a/b = \infty)$ | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

*Panel c*

| | $F_{III}$ | | | | | | | | $F^*_{III}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mu_2/\mu_1 = 0$ | $\beta$ | $\mu_2/\mu_1 = 0.5$ | $\beta$ | $\mu_2/\mu_1 = 2.0$ | $\beta$ | $\mu_2/\mu_1 = \infty$ | $\beta$ | $\mu_2/\mu_1 = 0$ | $\beta$ | $\mu_2/\mu_1 = 0.5$ | $\beta$ | $\mu_2/\mu_1 = 2.0$ | $\beta$ | $\mu_2/\mu_1 = \infty$ | $\beta$ |
| $h/2b = 0.1$ | | | | | | | | | | | | | | | | |
| 1 | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | |
| 2 | 0.1547 | 38 | 0.0061 | 30 | −0.0052 | 33 | −0.0146 | 33 | 0.0977 | 38 | 0.0039 | 30 | −0.0033 | 33 | −0.0092 | 33 |
| 4 | 0.1453 | 29 | 0.0098 | 19 | −0.0079 | 21 | −0.0222 | 22 | 0.0772 | 29 | 0.0052 | 19 | −0.0042 | 21 | −0.0118 | 22 |
| 16 | 0.0484 | 17 | 0.0079 | 9 | −0.0064 | 10 | −0.0183 | 10 | 0.0229 | 17 | 0.0037 | 9 | −0.0030 | 10 | −0.0087 | 10 |
| $\to\infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/$ $(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = 0.2$ | | | | | | | | | | | | | | | | |
| 1 | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | |
| 2 | 0.1019 | 41 | 0.0081 | 33 | −0.0058 | 33 | −0.0155 | 34 | 0.0644 | 41 | 0.0051 | 33 | −0.0037 | 33 | −0.0098 | 34 |
| 4 | 0.1003 | 34 | 0.0117 | 24 | −0.0088 | 24 | −0.0241 | 24 | 0.0410 | 34 | 0.0062 | 24 | −0.0048 | 24 | −0.0129 | 24 |
| 16 | 0.0329 | 22 | 0.0069 | 14 | −0.0059 | 13 | −0.0169 | 13 | 0.0156 | 22 | 0.0032 | 14 | −0.0028 | 13 | −0.0080 | 13 |
| $\to\infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/$ $(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = 0.3$ | | | | | | | | | | | | | | | | |
| 1 | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | |
| 2 | 0.0729 | 44 | 0.0900 | 38 | −0.0063 | 37 | −0.0162 | 37 | 0.0461 | 44 | 0.0057 | 38 | −0.0040 | 37 | −0.0102 | 37 |
| 4 | 0.0771 | 38 | 0.0118 | 30 | −0.0090 | 29 | −0.0243 | 29 | 0.0410 | 38 | 0.0063 | 30 | −0.0048 | 29 | −0.0129 | 29 |
| 16 | 0.0261 | 20 | 0.0066 | 15 | −0.0058 | 14 | −0.0168 | 14 | 0.0123 | 20 | 0.0031 | 15 | −0.0027 | 14 | −0.0079 | 14 |
| $\to\infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/$ $(a/b = \infty)$ | | | | | | | | | | | | | | | | |

Table 5 (*continued*)

| $a/b$ | $F_{\mathrm{III}}$ | | | | | | | | $F^*_{\mathrm{III}}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mu_2/\mu_1 = 0$ | $\beta$ | $\mu_2/\mu_1 = 0.5$ | $\beta$ | $\mu_2/\mu_1 = 2.0$ | $\beta$ | $\mu_2/\mu_1 = \infty$ | $\beta$ | $\mu_2/\mu_1 = 0$ | $\beta$ | $\mu_2/\mu_1 = 0.5$ | $\beta$ | $\mu_2/\mu_1 = 2.0$ | $\beta$ | $\mu_2/\mu_1 = \infty$ | $\beta$ |
| $h/2b = 0.4$ | | | | | | | | | | | | | | | | |
| 1 | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | |
| 2 | 0.0539 | 45 | 0.0087 | 41 | −0.0062 | 40 | −0.0159 | 40 | 0.0340 | 45 | 0.0055 | 41 | −0.0039 | 40 | −0.0100 | 40 |
| 4 | 0.0621 | 41 | 0.0111 | 34 | −0.0086 | 33 | −0.0234 | 33 | 0.0330 | 41 | 0.0059 | 34 | −0.0046 | 33 | −0.0124 | 33 |
| 16 | 0.0212 | 26 | 0.0056 | 19 | −0.0050 | 17 | −0.0147 | 18 | 0.0100 | 26 | 0.0026 | 19 | −0.0024 | 17 | −0.0069 | 18 |
| $\to \infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = 0.5$ | | | | | | | | | | | | | | | | |
| 1 | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | | 0.0000 | |
| 2 | 0.0404 | 46 | 0.0077 | 44 | −0.0057 | 44 | −0.0147 | 42 | 0.0255 | 46 | 0.0049 | 44 | −0.0036 | 44 | −0.0093 | 42 |
| 4 | 0.0512 | 43 | 0.0102 | 38 | −0.0080 | 37 | −0.0218 | 36 | 0.0272 | 43 | 0.0054 | 38 | −0.0043 | 37 | −0.0115 | 36 |
| 16 | 0.0178 | 28 | 0.0047 | 24 | −0.0043 | 23 | −0.0128 | 23 | 0.0084 | 28 | 0.0022 | 24 | −0.0020 | 23 | −0.0060 | 23 |
| $\to \infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = 1.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| 2 | 0.01104 | 45 | 0.00276 | 45 | −0.00228 | 45 | −0.00608 | 45 | 0.00695 | 45 | 0.00175 | 45 | −0.00144 | 45 | −0.00384 | 45 |
| 4 | 0.02243 | 48 | 0.00554 | 47 | −0.00462 | 46 | −0.02695 | 44 | 0.01191 | 48 | 0.00294 | 47 | −0.00246 | 46 | −0.01112 | 44 |
| 16 | 0.00998 | 36 | 0.00280 | 35 | −0.00262 | 35 | −0.00799 | 36 | 0.00472 | 36 | 0.00132 | 35 | −0.00124 | 35 | −0.00378 | 36 |
| $\to \infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = 2.0$ | | | | | | | | | | | | | | | | |
| 1 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| 2 | 0.00146 | 43 | 0.00039 | 42 | −0.00033 | 42 | −0.00089 | 42 | 0.00092 | 43 | 0.00024 | 42 | −0.00021 | 42 | −0.00056 | 42 |
| 4 | 0.00564 | 46 | 0.00150 | 46 | −0.00129 | 46 | −0.00351 | 45 | 0.00300 | 46 | 0.00080 | 46 | −0.00068 | 46 | −0.00186 | 45 |
| 16 | 0.00499 | 45 | 0.00142 | 45 | −0.00132 | 45 | −0.00395 | 45 | 0.00236 | 45 | 0.00067 | 45 | −0.00062 | 45 | −0.00187 | 45 |
| $\to \infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/(a/b = \infty)$ | | | | | | | | | | | | | | | | |
| $h/2b = \infty$ | | | | | | | | | | | | | | | | |
| 1 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| 2 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| 4 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| 16 | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | | 0.00000 | |
| $\to \infty$ | | | | | | | | | | | | | | | | |
| $(a/b = 1)/(a/b = \infty)$ | | | | | | | | | | | | | | | | |

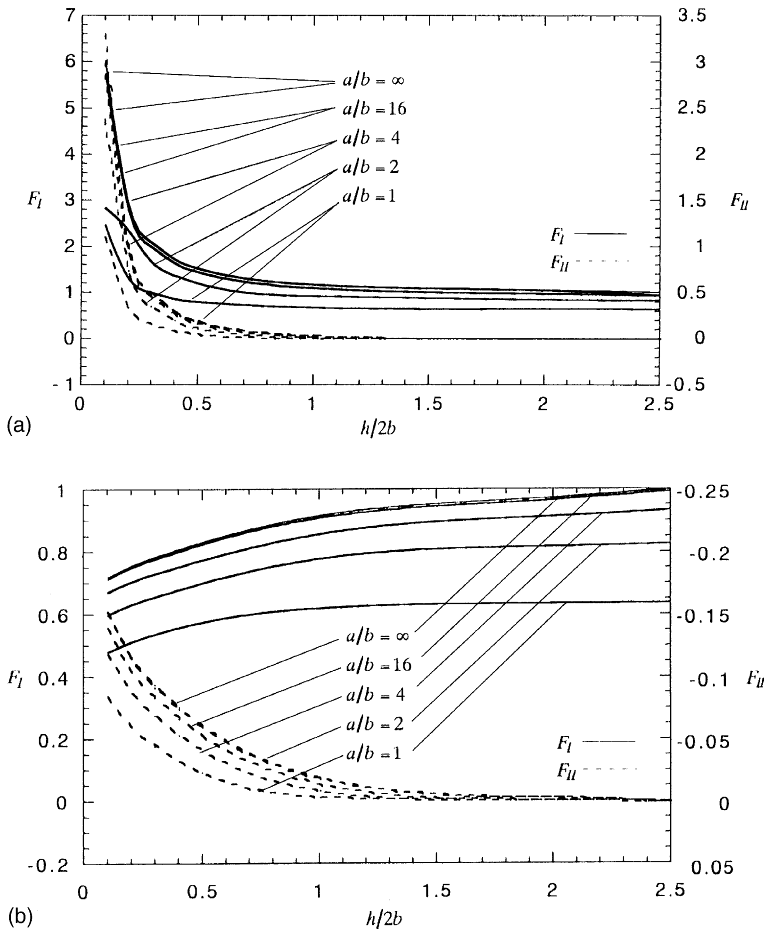
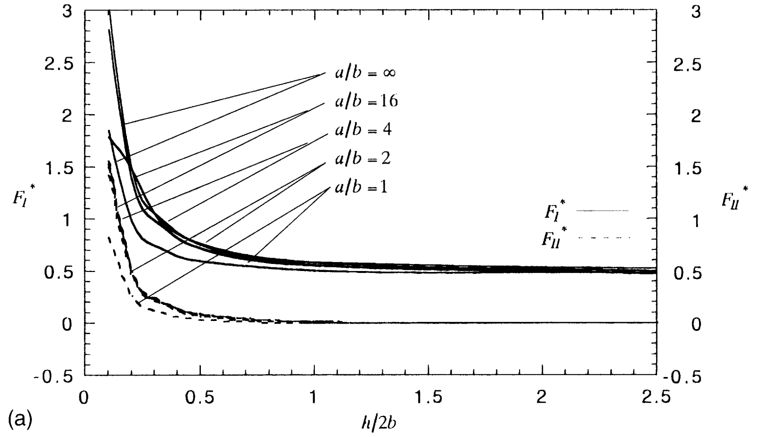



Fig. 5. (a) Variation of $F_I$, $F_{II}$ in Fig. 1 when $\mu_2/\mu_1 = 0$, $\nu_1$, $\nu_2 = 0.3$. (b) Variation of $F_I$, $F_{II}$ in Fig. 1 when $\mu_2/\mu_1 = \infty$, $\nu_1$, $\nu_2 = 0.3$.

(1) The problem is formulated as a system of singular integral equations correctly. In the numerical calculation, fundamental density functions and polynomials are used to approximate unknown body force densities. The results show that the present method have convergence to the fourth digit when $a/b =$ 1–16 and $h/2b \geqslant 0.1$ in Fig. 1 (see Tables 1 and 2).

(2) The stress intensity factors are indicated in tables and figures with varying the shape of crack $a/b = 1$–$\infty$, distance from the interface $h/2b = 0.1$–$\infty$, and elastic constants $\mu_2/\mu_1 =$ 0–$\infty$ when $\nu_1$, $\nu_2 = 0.3$ (see Table 5). The effect of Poisson's ratio is not very large, i.e. by about 11% when $a/b = 16$, $h/2b = 0.4$ and by about 5% when $a/b = 1$, $h/2b = 0.4$.

(3) The $\sqrt{\text{area}}$ parameter $F_I^*$ is found to be effective for engineering use because the effect of crack shape $a/b$ on $F_I^*$ is small. In other words, different shaped cracks have almost the same values of $F_I^*$ (see Figs. 5 and 6 and Table 5). The maximum values of $F_I$, $F_{II}$ appearing at $\beta = \pi/2$ becomes greatly influenced by the interface according to $h/2b \to 0$ especially for large value of $a/b$ (see Figs. 7–9).



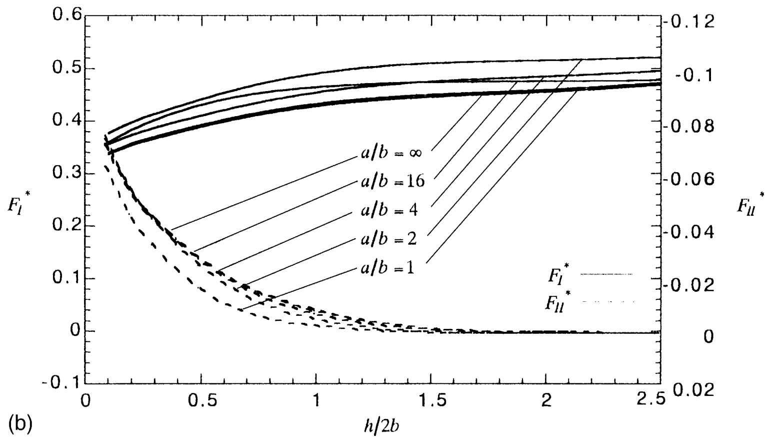




Fig. 6. (a) Variation of $F_{\rm I}^*$, $F_{\rm II}^*$ in Fig. 1 when $\mu_2/\mu_1 = 0$, $\nu_1$, $\nu_2 = 0.3$. (b) Variation of $F_{\rm I}^*$, $F_{\rm II}^*$ in Fig. 1 when $\mu_2/\mu_1 = \infty$, $\nu_1$, $\nu_2 = 0.3$.

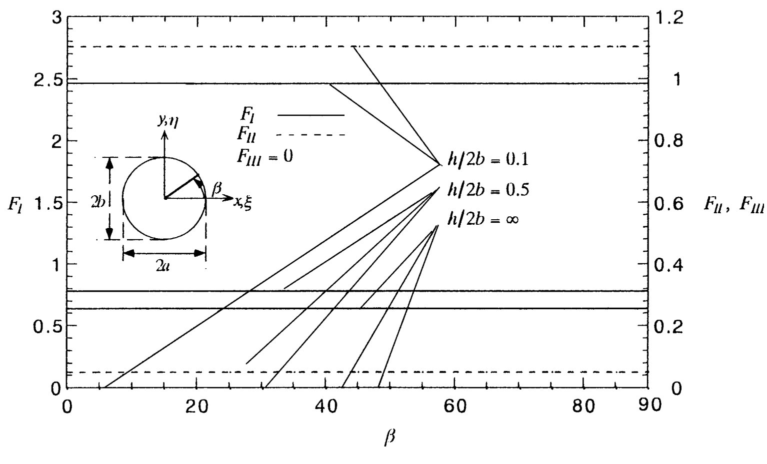



Fig. 7. Variation of $F_{\rm I}$, $F_{\rm II}$, $F_{\rm III}$ in Fig. 1 when $a/b = 1$, $\mu_2/\mu_1 = 0$, $\nu_1$, $\nu_2 = 0.3$.

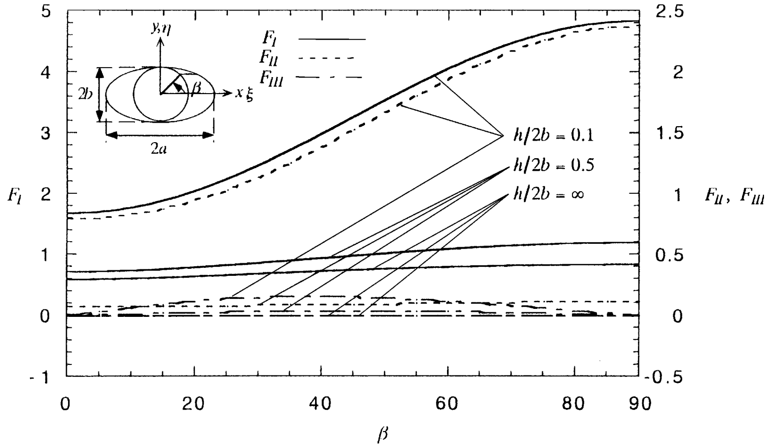



Fig. 8. Variation of $F_{\rm I}$, $F_{\rm II}$, $F_{\rm III}$ in Fig. 1 when $a/b = 2$, $\mu_2/\mu_1 = 0$, $\nu_1$, $\nu_2 = 0.3$.

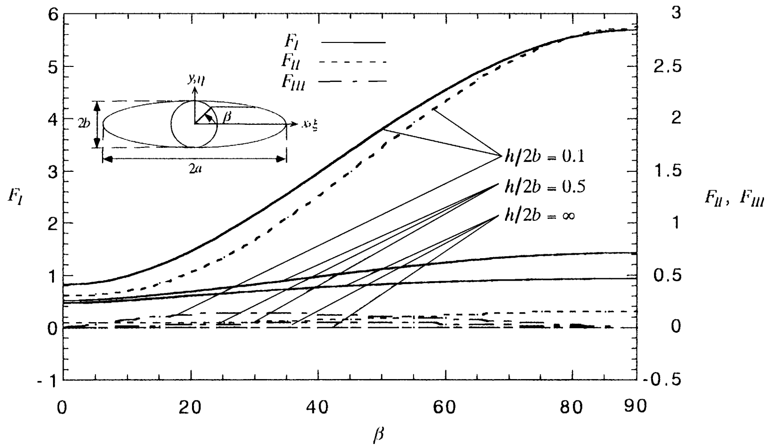



Fig. 9. Variation of $F_{\rm I}$, $F_{\rm II}$, $F_{\rm III}$ in Fig. 1 when $a/b = 4$, $\mu_2/\mu_1 = 0$, $\nu_1$, $\nu_2 = 0.3$.

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