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An Iterative Algorithm of Hypersingular Integral Equations for Crack Problems

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Abstract. Crack problems are reducible to singular integral equations with strongly singular kernels by means of the body force method. In the ordinary method, the integral equations are reduced to a system of linear algebraic equations. In this paper, an iterative method for the numerical solution of the hypersingular integral equations of the body force method is proposed. This method is based on the Gauss- Chebyshev numerical integration rule and is very simple to program. The solution is achieved without solving the system of linear algebraic equations. The proposed method is applied to some plane elasticity crack problems and is seen to give convergent results.

Introduction

The body force method (BFM) [1] has been widely applied to the analysis of the stress concentration factors and the stress intensity factors for various notch and crack problems. In the analysis of the method, two-dimensional crack problem is formulated as an integral equation with higher singularity of the form r^{-2}. The singular integral equation is called the hypersingular integral equation and can be accurately analyzed by using the fundamental density function and the Chebyshev polynomials [4]. In order to analyze this equation, it is necessary to perform a numerical integration of kernel function and a numerical solution of linear algebraic equation. Recently, the versatile program based on the body force method is developed to satisfy the both conditions of accuracy and efficiency [2]. This program contains no numerical integration and uses the boundary condition expressed by resultant force.

As the different way of the efficient computation, Ioakimidis [3] has considered the iterative method for numerical solution of singular integral equation with the Cauchy type kernel. The iterative algorithm is useful for the solution of the equation and the determination of stress intensity factor.

In this paper, a simple iterative algorithm for solving the hypersingular integral equation of the body force method is proposed and is applied to some crack problems. The solutions are achieved without solving system of linear algebra equations.

Method of Analysis

In the body force method, the problem of mode I crack is expressed by the following integral equation having a strongly singular kernel [4].

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)}{(t-x)^2}g(t)dt+\frac{1}{\pi}\int_{-1}^{1}w(t)h(t,x)g(t)dt=-p(x),\quad(-1<x<1)\,, \tag{1}$$

where $w(t)=\sqrt{1-t^2}$ is the weight function [1, 4], *g(t)* is the unknown function (proportional to the density of force doublets along the crack [1, 4]), *h(t,x)* is a regular kernel and *p(x)* is a normal traction on the crack surface. The first integral in (1) should be interpreted as a Hadamard finite-part integral.

The ordinary body force method has been used to reduce eq.(1) to a system of linear algebraic equations [1, 2, 4]. In this study, the iterative method for the numerical solution of eq.(1) is proposed.

Gauss-Chebyshev Quadrature Formula. Korsunsky [5] has proposed the following quadrature formula for the finite-part integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)}{(t-x)^2}g(t)dt \cong \sum_{i=1}^{m}\frac{(1-t_i^2)}{(m+1)}\frac{g(t_i)}{(t_i-x_j)^2}-(m+1)g(x_j)\,, \tag{2}$$

The nodes t_i and x_j are given by

$$t_i=\cos\left(\frac{i\pi}{m+1}\right),\quad x_j=\cos\left(\frac{(2j-1)\pi}{2(m+1)}\right),\quad (i=1,\ldots,m,\ j=1,\ldots,m+1)\,, \tag{3}$$

where t_i, ($i=1,\ldots,m$) and x_j, ($j=1,\ldots,m+1$) are zeros of $U_m(t_i)$ and $T_{m+1}(x_j)$, the Chebyshev polynomials of the first and second kinds, respectively [5].

Since the Korsunsky's integration formula (2) is valid only at the discrete set of points $x=x_j$, we derive the modified integration rule from the relation between the Cauchy principal-value integral and the hypersingular integral as follows [6],

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)g(t)}{(t-x)^2}dt \cong \sum_{i=1}^{m}\left\{\frac{1-t_i^2}{(m+1)(t_i-x)^2}\left(1-\frac{T_{m+1}(x)}{T_{m+1}(t_i)}\right)-\frac{1-t_i^2}{t_i-x}\frac{U_m(x)}{T_{m+1}(t_i)}\right\}g(t_i)\,. \tag{4}$$

Moreover, by using the Lagrangian interpolation (5), we obtain the integration formula (6).

$$g(x)=\sum_{i=1}^{m}\frac{U_m(x)}{U_m'(t_i)(x-t_i)}g(t_i)\,,\quad U_m'(x)=\frac{xU_m(x)-(m+1)T_{m+1}(x)}{1-x^2}\,, \tag{5}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)g(t)}{(t-x)^2}dt \cong \sum_{i=1}^{m}\frac{1-t_i^2}{(m+1)(t_i-x)^2}\left(1-\frac{T_{m+1}(x)}{T_{m+1}(t_i)}\right)g(t_i)-(m+1)g(x)\,. \tag{6}$$

It is seen that (4) and (6) are considered as the more general integration formula for hypersingular integrals which is valid at any points *x*. Clearly, (6) coincides with the Korsunsky's integration formula (2) when $x=x_j$. In the following section, we propose two methods of iterative algorithm using the formulae (2) and (6).

Iterative Method (A). Substituting from (2) into (1), eq.(1) becomes

$$(m+1)g_k(x_j)=p(x_j)+\sum_{i=1}^{m}\frac{1-t_i^2}{(m+1)}\left\{\frac{1}{(t_i-x_j)^2}+h(t_i,x_j)\right\}g_k(t_i),\quad (k=0,1,2,\ldots)\,, \tag{7}$$

where the nodes t_i and x_j are defined by (3). Eq.(7) is not sufficient for the construction of the algorithm because both values of $g_k(t_i)$ and $g_k(x_j)$ are involved. Hence, we use the Lagrangian interpolation to calculate the values $g_{k+1}(t_i)$,

$$g_{k+1}(t_i)=\sum_{j=1}^{m+1}\frac{T_{m+1}(x_j)}{T_{m+1}'(x_j)(t_i-x_j)}g_k(x_j)\,,\quad T_{m+1}'(x)=(m+1)U_m(x)\,. \tag{8}$$

Eqs. (7) and (8) describe the iterative algorithm (A) proposed in this study. The values $g_k(x_j)$ are calculated from (7) when the values $g_k(t_i)$ are given, and the values $g_{k+1}(t_i)$ are determined from (8). These values can then be used for the next iteration.

After the determination of the convergent values $g_k(t_i)$ and $g_k(x_j)$, the stress-intensity factors $K_I(\pm 1)$ are determined by the interpolation method in [7].

$$K_I(\pm 1) = g_k(\pm 1)\sqrt{\pi a}, \quad g_k(\pm 1) = \frac{1}{m}\sum_{j=1}^{m+1} U_{2m-2}\left((\frac{1 \pm x_j}{2})^{\frac{1}{2}} \right) g_k(x_j) \ . \tag{9}$$

In (9), the notation '*a*' means a half crack length.

Iterative Method (B). Next, we consider different algorithm using the integration formula (6) because the values $g_k(\pm 1)$ at the crack tip cannot be directly calculated in the method (A). By applying (6) at the points $x = t_j$ to (1), we obtain

$$(m+1)g_k(t_j) = p(t_j) + \sum_{i=1}^{m} \frac{1-t_i^2}{(m+1)}\left(1 - \frac{T_{m+1}(t_j)}{T_{m+1}(t_i)}\right)\left\{\frac{1}{(t_i - t_j)^2} + h(t_i, t_j)\right\} g_k(t_i), \tag{10}$$

where the points t_j are defined by

$$t_j = \cos\{i\pi/(m+2)\}, \quad (j = 0,1,\ldots,m+2)\ . \tag{11}$$

To calculate the next values $g_{k+1}(t_i)$ from the values $g_k(t_j)$ in (10), we use (6) once more with degree *m*+1 at the points $x = t_j$. Then,

$$(m+2)g_{k+1}(t_i) = p(t_i) + \sum_{j=1}^{m+1} \frac{1-t_j^2}{(m+2)}\left(1 - \frac{T_{m+2}(t_i)}{T_{m+2}(t_j)}\right)\left\{\frac{1}{(t_j - t_i)^2} + h(t_i, t_j)\right\} g_k(t_j)\ . \tag{12}$$

Eqs. (10) and (12) describe completely the iterative method (B). The calculation of (10) and (12) are repeated until the values $g_k(t_i)$ and $g_k(t_j)$ converge. The stress-intensity factors $K_I(\pm 1)$ are directly obtained from the crack-tip values $g_k(\pm 1)$ when *j*=0 and *m*+2 in (10).

Results and Discussion

In this section, we apply the iterative methods (A) and (B) of the previous section to mode I crack problems. In all problems the initial value for $g_0(t_i)$ in (7) and (10) is selected as the same value of the traction $p(t_i)$, that is $g_0(t_i) = p(t_i)$.

Two Collinear Cracks. As a first application, consider the problem of two collinear cracks with length *2a* in an infinite plane. In this case

$$h(t,x) = 1/(d - t - x)^2, \quad p(x) = 1, \tag{13}$$

where *d* is the distance between centre of cracks. Numerical results are presented in Tables 1 and 2 when a/d=0.25. The exact solution of the problem is analyzed by Erdogan [8]. As shown in tables, numerical results obtained by both proposed methods are in remarkable agreement with the theoretical values. The present iterative methods give rapidly converging numerical results even when *m*=3.

Table 1 Convergence of SIFs for two collinear cracks with increasing iteration *k* [method (A)]

Method (A)	m=3		m=4		m=5		m=6	
k	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)
10	1.0456	1.0257	1.0436	1.0239	1.0414	1.0220	1.0393	1.0202
20	1.0478	1.0278	1.0475	1.0275	1.0469	1.0269	1.0461	1.0261
30	1.0479	1.0279	1.0479	1.0279	1.0478	1.0278	1.0475	1.0275
40	1.0479	1.0280	1.0480	1.0279	1.0479	1.0279	1.0479	1.0279
Erdogan [8]	1.0480	1.0280	1.0480	1.0280	1.0480	1.0280	1.0480	1.0280

Table 2 Convergence of SIFs for two collinear cracks with increasing iteration *k* [method (B)]

Method (B)	m=3		m=4		m=5		m=6	
K	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)
10	1.0470	1.0275	1.0471	1.0273	1.0465	1.0266	1.0456	1.0258
20	1.0473	1.0278	1.0478	1.0280	1.0478	1.0279	1.0478	1.0278
30	1.0473	1.0278	1.0478	1.0280	1.0479	1.0279	1.0479	1.0279
40	1.0473	1.0278	1.0478	1.0280	1.0479	1.0280	1.0480	1.0280
Erdogan [8]	1.0480	1.0280	1.0480	1.0280	1.0480	1.0280	1.0480	1.0280

Table 3 Convergence of SIFs for an internal crack in a semi-infinite plate with increasing *k* [method (A)]

Method (A)	m=3		m=4		m=5		m=6	
k	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)
10	1.1247	1.3561	1.1109	1.3373	1.0986	1.3171	1.0880	1.2983
20	1.1443	1.3836	1.1402	1.3795	1.1347	1.3715	1.1284	1.3622
30	1.1464	1.3865	1.1454	1.3868	1.1436	1.3840	1.1411	1.3801
40	1.1466	1.3868	1.1463	1.3880	1.1458	1.3869	1.1448	1.3854
50	1.1466	1.3869	1.1464	1.3882	1.1463	1.3876	1.1460	1.3870
Noda et al [9]	1.1464	1.3875	1.1464	1.3875	1.1464	1.3875	1.1464	1.3875

Table 4 Convergence of SIFs for an internal crack in a semi-infinite plate with increasing *k* [method (B)]

Method (B)	m=5		m=6		m=7		m=8	
k	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)	g(1)	g(-1)
10	1.1317	1.3617	1.1250	1.3553	1.1185	1.3467	1.1118	1.3370
20	1.1450	1.3803	1.1439	1.3820	1.1425	1.3812	1.1404	1.3789
30	1.1459	1.3816	1.1458	1.3847	1.1458	1.3859	1.1454	1.3858
40	1.1460	1.3817	1.1460	1.3850	1.1463	1.3865	1.1462	1.3870
50	1.1460	1.3817	1.1461	1.3850	1.1463	1.3866	1.1464	1.3872
Noda et al [9]	1.1464	1.3875	1.1464	1.3875	1.1464	1.3875	1.1464	1.3875

Internal Crack in a Semi-Infinite Plate. We also considered the problem of internal crack with length *2a* in a semi-infinite plane. In this case

$$h(t,x) = -\frac{1}{(t+x+2d)^2} + \frac{12(x+d)}{(t+x+2d)^3} - \frac{12(x+d)^2}{(t+x+2d)^4}, \quad p(x) = 1, \tag{14}$$

where *d* is the distance between the center of crack and the free edge of the semi-infinite plate. Numerical results are shown in Tables 3 and 4 when a/d=0.8. The accurate stress-intensity factor of the problem is analyzed by the singular integral equation method [9]. As shown in tables, in spite of this inconvenient geometry condition, numerical results are in close agreement with the previous results in [9]. From Table 4, in method (B) larger number of *m* (compared with those used in method (A)) must be used to obtain the same accuracy in the final numerical results.

From Tables 1-4, the present iterative methods were found to give the good convergence of the numerical results and to be useful for determining the stress intensity factor.

References

[1] H. Nisitani: *Journal of the Japan Soc. Mech. Engng.*, Vol.70, No.580 (1967), p.627.

[2] H. Nisitani, D.-H. Chen and A. Saimoto: *Versatile Program of Two-dimensional Stress Analysis Based on Body Force Method* (Baifukan Publications, Japan 1994).

[3] N.I. Ioakimidis: *Journal of Computational Physics*, Vol. 43 (1981), p. 164.

[4] N.-A. Noda and K. Oda: *International Journal of Fracture*, Vol.58, No. 4 (1992), p. 285.

[5] A.M. Korsunsky: *Quarterly of Applied Mathematics*, Vol. LVI, No. 3 (1998), p. 461.

[6] G. Monegato: *Numer. Math.*, Vol. 50 (1987), p. 273.

[7] S. Krenk: *Quarterly of Applied Mathematics*, Vol. 32 (1975), p. 479.

[8] F. Erdogan: *Proc. 4th U.S. Nat. Congr. Appl. Mech.*, (1962), p. 547.

[9] N.-A. Noda and T. Matsuo: *International Journal of Fracture*, Vol. 63, No. 3 (1993), p. 229.

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