Proceedings of the 8th International Congress on Thermal Stresses, TS2009
1–4 June, University of Illinois at Urbana-Champaign

Analysis of stress intensity factors of interface crack under polynomial distribution of stress

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In this paper, the stress intensity factors for a two-dimensional interfacial crack under polynomial distribution of stress are considered on the basis of the body force method. In the numerical calculations, unknown body force densities are approximated by the products of the fundamental densities and power series; here the fundamental densities are chosen to express singular stress fields due to an interface crack exactly. The calculation shows that the present method gives exact solutions under polynomial distribution of stress.

Key Words : *Stress intensity factor, Body force method, Interface crack, Singular integral equation.*

1 Introduction

Composite materials and adhesive or bonded joints are being used in wide range of engineering field in recent years, and the interface crack is been paid much attention gradually. Since little solutions are available for arbitrary material combination, in this paper stress intensity factors are considered for 2D interface crack under polynomial distribution of stress. The problem will be formulated on the basis of singular integral equations and the solutions will be discussed. The present method of analysis can be applied to 3D interface crack problems.

When a 2D homogeneous crack with the length of 2a in an infinite plate is subjected to the constant stress $\sigma_y = \sigma_0, \tau_{xy} = \tau_0, \tau_{yz} = \tau_0$, the stress intensity actors (SIFs) K_I, K_{II}, K_{III} are known as

$$K_I = \sigma_0\sqrt{\pi a}\ ,\ \ K_{II} = \tau_0\sqrt{\pi a}\ , K_{III} = \tau_0\sqrt{\pi a}\ . \qquad (1a)$$

More generally, if the crack is subjected to general stress which can be expressed as the polynomial distribution $p_y = p_0\sum_{n=0}^{\infty}\alpha_n(x/a)^n$, the stress intensity factors are given in the following forms [1].

$$\begin{aligned} K_{I,A} &= p_0\sqrt{\pi a}[\alpha_0 + \sum_{n=1}^{\infty}(\alpha_{2n-1}+\alpha_{2n})(2n)!/2^{2n}(n!)^2] \\ K_{I,B} &= p_0\sqrt{\pi a}[\alpha_0 + \sum_{n=1}^{\infty}(-\alpha_{2n-1}+\alpha_{2n})(2n)!/2^{2n}(n!)^2] \end{aligned} \qquad (1b)$$

Using Eq. (1) Table 1 indicates the value of dimensionless stress intensity factors with varying the exponent n when the crack is subjected to the polynomial distribution of stress $p_y = p_0(x/a)^n$.

On the other hand, when a 2D interface crack is subjected to the constant pressure, the stress intensity factors are given in the following expression [2]:

$$K_1 + iK_2 = (\sigma_y^{\infty} + i\tau_{xy}^{\infty})(1+2i\varepsilon)\sqrt{\pi a}\ . \qquad (2)$$

Here the bimaterial constant ε is defined as $\varepsilon = 1/2\pi\ln\{(\mu_2\kappa_1+\mu_1)/(\mu_1\kappa_2+\mu_2)\}$ where the elastic constants are given as shear modulus and Poisson ratio for the upper (material 1) and the lower (material 2) half-planes, that is (μ_1, ν_1) and (μ_2, ν_2).

$$\kappa_{1,2} = \begin{cases} (3-\nu_{1,2})/(1+\nu_{1,2}) & \text{(plane stress)} \\ 3-4\nu_{1,2} & \text{(plane strain)} \end{cases} \qquad (3)$$

Table 1: Stress intensity factors of homogeneous crack under polynomial distribution of stress.
$F_{I,A} = K_{I,A}/p_0\sqrt{\pi a}, F_{I,B} = K_{I,B}/p_0\sqrt{\pi a}$,

n	$F_{I,A}$	$F_{I,B}$
0	1.0	1.0
1	0.5	-0.5
2	0.5	0.5
3	0.375	-0.375
4	0.375	0.375
5	0.3125	-0.3125
6	0.3125	0.3125

However, the stress intensity factors for the 2D interface crack which is subjected to more general distribution of stress have not been provided yet in the references and handbooks. Therefore in this paper the stress intensity factors for the 2D interface crack which is subjected to polynomial distribution of stress have been analyzed on the basis of the body force method coupled with singular integral equation formulation.

The body force method was originally proposed by Nisitani as a new method for solving concentration problems. In solving crack problems, the body force method uses the stress fields due to a pair of point forces or crack opening displacements [3]. In those analyses, the problems may be formulated as a system of singular equations [4].

2 Numerical solution of the interface crack

Singular integral equations for two-dimensional cracks on a bimaterial interface can be expressed in Eq.(4). Here, (ξ,η) is a (x,y) coordinate where body forces are applied.

$$-\pi\beta\frac{dP_{yx}(x)}{dx}+\int_{-a}^{a}\frac{P_{yy}(\xi)}{(\xi-x)^2}d\xi+\sum\int_{-a}^{a}h_y(\xi,x)P_{yy}(\xi)d\xi$$
$$=-\sum_{m=1}^{2}\frac{\mu_m(1+\kappa_m)}{\kappa_m-1}\frac{\pi}{C}p_y$$
$$\pi\beta\frac{dP_{yy}(x)}{dx}+\int_{-a}^{a}\frac{P_{yx}(\xi)}{(\xi-x)^2}d\xi+\sum\int_{-a}^{a}h_y(\xi,x)P_{yx}(\xi)d\xi$$
$$=-\sum_{m=1}^{2}\frac{\mu_m}{C}p_x \tag{4}$$

$$\kappa_{1,2}=\begin{cases}(3-\nu_{1,2})/(1+\nu_{1,2}) & \text{(plane stress)}\\ 3-4\nu_{1,2} & \text{(plane strain)}\end{cases}$$

$$C=\frac{2\mu_1(1+\alpha)}{(1-\beta^2)(\chi_1+1)}=\frac{2\mu_2(1-\alpha)}{(1-\beta^2)(\chi_2+1)} \tag{5}$$

$$\alpha=\frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)},\beta=\frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}$$

In the numerical solution of equations (4), the unknown functions are approximated by the product of the fundamental density $F_1(\xi),F_2(\xi)$ and the weight functions $w_1(\xi),w_2(\xi)$ [4].

$$P_1(\xi)+iP_2(\xi)=\{w_1(\xi)+iw_2(\xi)\}\{F_1(\xi)+iF_2(\xi)\} \tag{6}$$

Here, the fundamental density functions are chosen to express the stress field of the crack, they are derived from the crack opening displacement of the interface crack and expressed as follows [5]:

$$\sum_{m=1}^{2}\left\{\frac{\kappa_m-1}{1+\kappa_m}w_{1j}(\xi_j)+iw_{2j}(\xi_j)\right\}=\sum_{m=1}^{2}\frac{1+\kappa_m}{4\cosh\pi\varepsilon}\sqrt{a_j^2-\xi_j^2}$$
$$\times\left(\frac{a_j-\xi_j}{a_j+\xi_j}\right)^{i\varepsilon} \tag{7}$$

In the numerical analysis, the weight functions $F_1(\xi),F_2(\xi)$ are approximated by the following power series:

$$F_1(\xi)=\sum_{n=1}^{M}a_n\xi^{n-1},F_2(\xi)=\sum_{n=1}^{M}b_n\xi^{n-1} \tag{8}$$

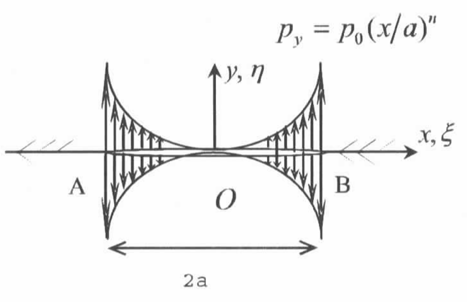

Fig.1: Two-dimensional interface crack under polynomial distribution of stress

3 Results and discussion

For general cases, the stress intensity factors can be defined as the follows:

$$\sigma_z+i\tau_{xz}\big|_{\theta=0}=\lim_{r\to 0}\frac{K_1+iK_2}{\sqrt{2\pi r}}\left(\frac{r}{l}\right)^{i\varepsilon}. \tag{9}$$

For the two-dimensional interfacial crack, by using the numerical method mentioned above, the stress intensity factors can be directly calculated from the value of the weight functions at the crack tip.

$$K_1+iK_2=p_0\times[(1+2i\varepsilon)\times\sqrt{\pi a}\times(F_1+iF_2)] \tag{10}$$

Table 2 shows the convergence of stress intensity factors of the two-dimensional interface cracks with varying the number of polynomial M the exponent n. It is seen that the results give exact values for the exponent n of the polynomial distribution of stress when $M\geq n+1$.

Besides when the interface is subjected to polynomial distribution of stress, the coefficients of the power functions have been examined. For example, when $\varepsilon=0.02$ and $\varepsilon=0.06$, the coefficients of the Eqs (8) a_i,b_i are shown in Table 3. It is found that when n=2N-1 (N is the natural number) $a_i\neq 0$, $b_i=0$ (i is even number and $i\leq n+1$), $a_i=0, b_i\neq 0$ (i is odd number and $i\leq n+1$), and $a_i=0$, $b_i=0$ (i>n+1). Also, when n=2N, $a_i\neq 0$, $b_i=0$ (i is odd and $i\leq n+1$), $a_i=0$, $b_i\neq 0$ (i is even number and $i\leq n+1$), and $a_j=0$, $b_j=0$ (i>n+1).

From the values of the coefficients for different values of ε as shown in Table 3, we have proposed the formula of the coefficients a_i,b_i as functions of ε, which are shown in the Table 4.

The crack opening displacements can be written as the following in the following way.

(1) When n=2N-1

$$\Delta u_y+i\Delta u_x=\sum_{m=1}^{2}\frac{1+\kappa_m}{4G_m\cosh\pi\varepsilon}\sqrt{a^2-\xi^2}\left(\frac{a-\xi}{a+\xi}\right)^{i\varepsilon}$$
$$\times\left\{c_1+c_3\xi^2+\cdots+c_{n+1}\xi^n+i\left(c_2'\xi+c_4'\xi^3+\cdots+c_n'\xi^{n-1}\right)\right\} \tag{11}$$

(2) When n=2N

$$\Delta u_y+i\Delta u_x=\sum_{m=1}^{2}\frac{1+\kappa_m}{4G_m\cosh\pi\varepsilon}\sqrt{a^2-\xi^2}\left(\frac{a-\xi}{a+\xi}\right)^{i\varepsilon}$$
$$\times\left\{c_2\xi+c_4\xi^3+\cdots+c_{n+1}\xi^n+i\left(c_1'+c_3'\xi^2+\cdots+c_n'\xi^{n-1}\right)\right\} \tag{12}$$

Table 5 shows the stress intensity factors of F_1 and F_2 when the two-dimensional interface crack is subjected to polynomial distribution of stress with varying the exponent $n=1\sim 6$. It is found that the results are controlled by bimaterial constant ε alone. In the Table 5 we have proposed the formula for F_1,F_2 when the interface crack is under polynomial distribution of tensile stress $p_y=p_0(x/a)^n$ and shear stress $p_x=p_0(x/a)^n$. The values in Table 5 have more than 4 digit accuracy.

Table 2: Convergence of results for dimensionless stress intensity factors for 2D interface crack under general internal pressure $p_y = p_0 (x/a)^n$ when ε =0.02. $K_1 + iK_2 = \{F_1 + iF_2\}\sqrt{\pi a}(1+2i\varepsilon)$

M	n=0		n=1		n=2		n=3		n=4		n=5		n=6	
	F_1	F_2	F_1	F_2	F_1	F_2	F_1	F_2	F_1	F_2	F_1	F_2	F_1	F_2
1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000
2	1.0000	0.0000	0.5000	0.0200	0.2500	0.0000	0.1250	0.00500	0.0625	0.0000	0.03125	0.00125	0.01563	0.00000
3	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.2222	0.00890	0.2221	0.0059	0.09877	0.00395	0.09872	0.00263
4	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.3748	0.01833	0.2772	0.0083	0.21668	0.01075	0.15567	0.00474
5	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.3748	0.01833	0.3747	0.0147	0.24865	0.01262	0.24855	0.01037
6	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.3748	0.01833	0.3747	0.0147	0.31222	0.01716	0.26921	0.01159
7	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.3748	0.01833	0.3747	0.0147	0.31222	0.01716	0.31212	0.01471
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
∞	1.0000	0.0000	0.5000	0.0200	0.4997	0.1333	0.3748	0.01833	0.3747	0.0147	0.31222	0.01716	0.31212	0.01471

Table 3: Coefficient a_n of power series

(a) when $\varepsilon = 0.02$

a_n	n=0	a_1=1
	n=1	a_2=0.5
	n=2	a_1=0.1664,a_3=0.3333
	n=3	a_2=0.1248,a_4=0.25
	n=4	a_1=0.07481,a_3=0.09984,a_5 =0.2
	n=5	a_2=0.06234,a_4=0.08320,a_6=0.1667
	n=6	a_1=0.04450,a_3=0.05344,a_5=0.07131,a_7=0.1429
b_n	n=0	b_0=0
	n=1	b_1=0.02
	n=2	b_2=0.01334
	n=3	b_1=0.008332,b_3=0.01
	n=4	b_2=0.006666,b_4=0.008001
	n=5	b_1=0.004943,b_3=0.005555,b_5=0.006668
	n=6	b_2=0.004237,b_4=0.004761,b_6=0.005715

(b) when $\varepsilon = 0.06$

a_n	n=0	a_1=1
	n=1	a_2=0.5
	n=2	a_1=0.1643,a_3=0.3333
	n=3	a_2=0.1232,a_4=0.25
	n=4	a_1=0.07332,a_3=0.09856,a_5=0.2000
	n=5	a_2=0.06110,a_4=0.08213,a_6=0.1667
	n=6	a_1=0.04339,a_3=0.05237,a_5=0.07040,a_7=0.1429
b_n	n=0	b_0=0
	n=1	b_1=0.06
	n=2	b_2=0.04
	n=3	b_1=0.02493,b_3=0.03
	n=4	b_2=0.01994,b_4=0.024
	n=5	b_1=0.01476,b_3=0.01662,b_5=0.02
	n=6	b_2=0.01265,b_4=0.01425,b_6=0.01714

Table 4: General expression for coefficient of power series

a_n	n=0	a_1=1
	n=1	a_2=0.5
	n=2	$a_1=-0.6667\varepsilon^2+0.16667$ a_3=0.333333
	n=3	$a_2=-0.5\varepsilon^2+0.125$ a_4=0.25
	n=4	$a_1=-0.46611\varepsilon^2+0.075$ $a_3=-0.4\varepsilon^2+0.1$ a_5=0.2
	n=5	$a_2=-0.388\varepsilon^2+0.0625$ $a_4=-0.3334\varepsilon^2+0.0833333$ a_6=0.166667
	n=6	$a_1=-0.35\varepsilon^2+0.0446428$ $a_3=-0.332\varepsilon^2+0.0535714$ $a_5=-0.28782\varepsilon^2+0.0714285$ a_7=0.142857
b_n	n=0	b_0=0
	n=1	$b_1=\varepsilon$
	n=2	$b_2=0.666667\varepsilon$
	n=3	$b_1=0.416667\varepsilon$ $b_3=0.5\varepsilon$
	n=4	$b_2=0.333333\varepsilon$ $b_4=0.4\varepsilon$
	n=5	$b_1=0.247135\varepsilon$ $b_3=0.27774\varepsilon$ $b_5=0.333333\varepsilon$
	n=6	$b_2=0.21183\varepsilon$ $b_4=0.238065\varepsilon$ $b_6=0.28575\varepsilon$

Table 5: Stress intensity factors F_1 and F_2 of 2D interface crack under general internal pressure.

$p_y = p_0(x/a)^n \quad K_1 + iK_2 = \{F_1 + iF_2\}\sqrt{\pi a}(1+2i\varepsilon)$

F_1	n=0	n=1	n=2	n=3	n=4	n=5	n=6
ε=0.00	1.00000	0.50000	0.50000	0.37500	0.37500	0.31250	0.31250
ε=0.02	1.00000	0.49999	0.49972	0.37479	0.37464	0.31220	0.31210
ε=0.04	1.00000	0.49999	0.49892	0.37419	0.37360	0.31133	0.31090
ε=0.06	1.00000	0.49999	0.49758	0.37319	0.37187	0.30989	0.30900
ε=0.08	1.00000	0.50000	0.49573	0.37180	0.36946	0.30788	0.30632
ε=0.10	1.00000	0.50000	0.49333	0.37000	0.36635	0.30530	0.30285

F_2	n=0	n=1	n=2	n=3	n=4	n=5	n=6
ε=0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
ε=0.02	0.00000	0.02000	0.01334	0.01833	0.01467	0.01716	0.01471
ε=0.04	0.00000	0.04000	0.02677	0.03655	0.02932	0.03430	0.02940
ε=0.06	0.00000	0.06000	0.04000	0.05493	0.04395	0.05138	0.04404
ε=0.08	0.00000	0.08000	0.05333	0.07316	0.05853	0.06838	0.05861
ε=0.10	0.00000	0.10000	0.06667	0.09142	0.07307	0.08528	0.07310

Table 6: General expression of stress intensity factors under polynomial distribution of stress

		F_1	F_2
$p_y = p_0(x/a)^n$	n=1	0.5	ε
	n=2	$0.5-0.6667\varepsilon^2$	$(2/3)\varepsilon$
	n=3	$0.375-0.5\varepsilon^2$	0.916ε
	n=4	$0.375-0.86611\varepsilon^2$	0.733ε
	n=5	$0.3125-0.7214\varepsilon^2$	0.857ε
	n=6	$0.3125-0.96982\varepsilon^2$	0.735ε
$p_x = p_0(x/a)^n$	n=1	ε	0.5
	n=2	$(2/3)\varepsilon$	$0.5-0.6667\varepsilon^2$
	n=3	0.916ε	$0.375-0.5\varepsilon^2$
	n=4	0.733ε	$0.375-0.86611\varepsilon^2$
	n=5	0.857ε	$0.3125-0.7214\varepsilon^2$
	n=6	0.735ε	$0.3125-0.96982\varepsilon^2$

4 Conclusions

In this study two-dimensional interface crack under polynomial distribution of stress is investigated through the singular integral equations on the basis of the body force method. This method can be applied to problems of three-dimensional interface cracks [6], and obviously it can also be applied to the problem under thermal stress. The conclusions of this paper can be summarized as follows.

(1) The unknowns of the singular integral equations for the interface crack are approximated by the product of the fundamental density function and the weight function. The fundamental density functions are chosen to express the stress field due to the interface crack, and the weight functions are approximated by the power series. Besides the coefficients of the power series have been analyzed, and the formulas of the coefficients with varying ε have been obtained.

(2) The results of the dimensionless stress intensity factors with varying value of the exponent of the polynomial distribution of stress have been obtained with varying the value of bimaterial constant ε. Besides, the stress intensity factors F_1, F_2 and crack opening displacements are given as formulas of ε when the interface crack is subjected to the polynomial distribution of stress.

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