

## Stress Intensity Factors of an Interface Crack under Polynomial Distribution of Stress\*

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### Abstract

In this paper, stress intensity factors for a two-dimensional interface crack under polynomial distribution of stress are considered. The problem is formulated as a system of hypersingular integral equations on the idea of the body force method. In this analysis, unknown body force densities are approximated by the products of the fundamental densities and power series; here the fundamental densities are chosen to express singular stress fields due to an interface crack under constant distribution of stress exactly. The stress intensity factors of a 2D interfacial crack under polynomial distribution of stress are expressed as formulas for the reader's convenience with the varying polynomial exponent  $n$ . The exact expressions of crack opening displacements are also indicated.

**Key words:** Elasticity, Stress Intensity Factor, Body Force Method, Interface Crack, Composite Material, Fracture Mechanics, Singular Integral Equation

### 1. Introduction

Dissimilar materials are found in a variety of important structures such as adhesive joints, composite laminates, and electronic and optic components. Due to the mismatch of the coefficients of the thermal expansion and the Young's modulus, singular stress and crack initiation often happen at the interfaces between the dissimilar materials, which result in the degradation of reliability of the structure. Hence, problem of interface crack in dissimilar materials is very important. Salganik<sup>[1]</sup> was the first to analyze the two-dimensional interface crack, and then many researchers<sup>[2]-[10]</sup> have also studied similar problems. However, most solutions are limited to the cases under certain material combinations. In other words, there are few solutions available for any combinations of materials. In this study, therefore, the interface crack will be considered under general loading and any combination of materials on the idea of the body force method.

When a two-dimensional crack with length  $2a$  is subjected to the polynomial distribution of stress  $p_{zz} = p_0 \sum_{n=0}^{\infty} \alpha_n (x/a)^n$ , the exact solution can be expressed by the following formula<sup>[9]</sup>.

$$\begin{aligned} K_{I,A} &= p_0 \sqrt{\pi a} \left[ \alpha_0 + \sum_{n=1}^{\infty} (\alpha_{2n-1} + \alpha_{2n}) (2n)! / 2^{2n} (n!)^2 \right] \\ K_{I,B} &= p_0 \sqrt{\pi a} \left[ \alpha_0 + \sum_{n=1}^{\infty} (-\alpha_{2n-1} + \alpha_{2n}) (2n)! / 2^{2n} (n!)^2 \right] \end{aligned} \quad (1)$$

The formula in Eq.(1) is often used to evaluate cracks in residual stress or general stress fields. For the interface crack, however, the solution corresponding to the formula (1) is not

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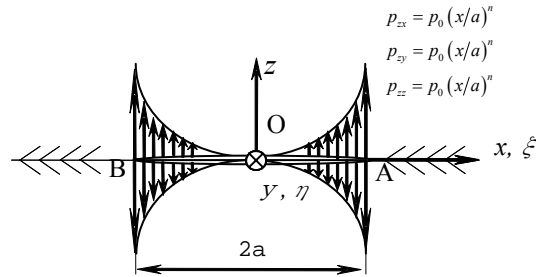


Fig.1 Two-dimensional interface crack under polynomial distribution of stress

known. Therefore, in this paper, general expressions of the stress intensity factor have been considered for an interface crack under the polynomial distribution of stress with varying material constants. The solution considered in this paper may be also useful for analyzing three-dimensional interface crack problems [11].

## 2. Numerical solutions

### 2.1 Solution for plane problem when the crack is subjected to $p_{zz}$ and $p_{zx}$ in Fig.1

As shown in Fig.1, stress intensity factors of interface crack under polynomial distribution of stress  $p_{zz} = p_0(x/a)^n$  will be analyzed. The stress distribution along an interface crack is expressed by the following equations around the crack tip.

$$\sigma_{zz} + i\tau_{zx} = \frac{K_I + iK_{II}}{\sqrt{2\pi r}} \left(\frac{r}{a}\right)^{ie} \quad (2)$$

$$\varepsilon = \frac{1}{2\pi} \ln \left\{ \frac{(G_2\kappa_1 + G_1)}{(G_1\kappa_2 + G_2)} \right\} \quad (3)$$

$$\kappa_m = \begin{cases} (3 - \nu_m)/(1 + \nu_m) & \text{(Plane stress)} \\ 3 - 4\nu_m & \text{(Plane strain)} \end{cases}$$

$\nu_m$ : Poisson's ratio ( $m=1,2$ )

$G_m$ : Shear modulus ( $m=1,2$ )

The problem can be formulated in terms of singular integral equations by using the stress fields at the interface when two kinds of standard set of force doublets, tension type and shear type, act on a point of interface [6], [10]. The integral equations, which are virtually the boundary condition of the crack  $\sigma_z = 0, \tau_{zx} = 0$ , are expressed by the following equations.

$$-\pi\beta \frac{dP_2(x)}{dx} + \sum_{m=1}^2 \frac{\kappa_m - 1}{\kappa_m + 1} \int_{-a}^a \frac{P_1(\xi)}{(\xi - x)^2} d\xi = -\sum_{m=1}^2 G_m \frac{\pi}{C} p_{zz}(x) \quad (4)$$

$$\pi\beta \sum_{m=1}^2 \frac{\kappa_m - 1}{\kappa_m + 1} \frac{dP_1(x)}{dx} + \int_{-a}^a \frac{P_2(\xi)}{(\xi - x)^2} d\xi = -\sum_{m=1}^2 G_m \frac{\pi}{C} p_{zx}(x)$$

$$C = \frac{2G_1(1 + \alpha)}{(1 - \beta^2)(\kappa_1 + 1)} = \frac{2G_2(1 - \alpha)}{(1 - \beta^2)(\kappa_2 + 1)} \quad (5)$$

$$\alpha = \frac{G_2(\kappa_1 + 1) - G_1(\kappa_2 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}$$

$$\beta = \frac{G_2(\kappa_1 - 1) - G_1(\kappa_2 - 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)} \quad (6)$$

Here, the densities of body force doublets, tension type  $P_1(\xi)$  and shear type  $P_2(\xi)$  [10], which are distributed on the interface, are unknown functions, and  $x, \xi$  are the coordinate where body forces are applied.

On the other hand, the singular integral equations items of the crack opening displacement are expressed by the following equations.

$$\begin{aligned} -\beta \frac{d\Delta u_x(x)}{dx} + \frac{1}{\pi} \int_{-a}^a \frac{\Delta u_z(\xi)}{(\xi-x)^2} d\xi &= -\frac{p_z(x)}{C} \\ \beta \frac{d\Delta u_z(x)}{dx} + \frac{1}{\pi} \int_{-a}^a \frac{\Delta u_x(\xi)}{(\xi-x)^2} d\xi &= -\frac{p_x(x)}{C} \end{aligned} \quad (7)$$

The body force doublets and the crack opening displacements have the following relations.

$$\begin{aligned} P_1(\xi) &= -\sum_{m=1}^2 \frac{G_m(1+\kappa_m)}{\kappa_m-1} \Delta u_z(\xi) \\ P_2(\xi) &= \sum_{m=1}^2 G_m \Delta u_x(\xi) \end{aligned} \quad (8)$$

The index  $m = 1, 2$  represents the material 1,2, and  $\int$  denotes a finite-part integral. In the numerical solution of equation (4), the unknown functions  $P_1(\xi), P_2(\xi)$  are approximated by the product of the fundamental density  $w_1(\xi), w_2(\xi)$  and the weight functions  $F_1(\xi), F_2(\xi)$ .

$$P_1(\xi) + iP_2(\xi) = \{w_1(\xi) + iw_2(\xi)\} \{F_1(\xi) + iF_2(\xi)\} \quad (9)$$

By using the fundamental density function, the crack opening displacements are shown as the following equations [4],[10].

$$\begin{aligned} \Delta u_z + i\Delta u_x &= \sum_{m=1}^2 \frac{1}{G_m} \left\{ \frac{\kappa_m-1}{1+\kappa_m} w_1(\xi) + iw_2(\xi) \right\} (F_1(\xi) + iF_2(\xi)) \\ &= \sum_{m=1}^2 \frac{1}{G_m} \frac{1+\kappa_m}{4 \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a-\xi}{a+\xi} \right)^{i\varepsilon} (F_1(\xi) + iF_2(\xi)) \end{aligned} \quad (10)$$

Besides, the exact crack opening displacements can be expressed for the interface crack under constant internal pressure  $\sigma_0, \tau_0$  in Eq.(11).

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1+\kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a-\xi}{a+\xi} \right)^{i\varepsilon} (\sigma_0 + i\tau_0) \quad (11)$$

This formula can be used for analyzing under tensile and shear stresses. In this numerical analysis, the weight functions  $F_1(\xi), F_2(\xi)$  are approximated by the following power series.

$$F_1(\xi) = \sum_{n=1}^M a_n \xi^{n-1}, F_2(\xi) = \sum_{n=1}^M b_n \xi^{n-1} \quad (12)$$

A set of collocation points on the imaginary crack site is chosen as follows [6]:

$$x = a \cos \{n\pi / (M+1)\}, (n = 1, \dots, M), \quad (13)$$

where  $M$  is the number of the collocation points on the crack.

To satisfy the boundary condition, several collocation points should be placed as close to the ends of the cracks as shown in Eq.(13). Using the above method, the singular integral equations may be reduced to algebraic equations for the determination of coefficients  $a_n, b_n$ . The stress intensity factors are expressed from the unknown functions  $F_1(\xi), F_2(\xi)$  shown in Eq. (14).

$$K_1 + iK_2 = \{F_1(a) + iF_2(a)\} p_0 \sqrt{\pi a} (1 + 2i\varepsilon) \quad (14)$$

### 2.2 Solutions for transverse shear problem when a crack is subjected to $p_{zy}$ in Fig.1

The problem can be formulated in terms of singular integral equations by using the stress fields at the interface when the shear type of standard set of force doublets acts on a point of interface, The integral equations, which are virtually the boundary condition on the crack  $\tau_{zy} = 0$ , are expressed as follows.

$$\frac{1}{\pi} \frac{G_1 G_2}{G_1 + G_2} \int_{-a}^a \frac{P_3(\xi)}{(\xi - x)^2} d\xi = -\sum_{m=1}^2 G_m p_{zy}(x) \quad (15)$$

On the other hand, the singular integral equations of the crack displacement are expressed as the following equations.

$$\frac{1}{\pi} \frac{G_1 G_2}{G_1 + G_2} \int_{-a}^a \frac{\Delta u_y(\xi)}{(\xi - x)^2} d\xi = -p_{zy}(x) \quad (16)$$

Here we have the relation between the standard set of doublet and crack opening displacement as follows.

$$P_3(\xi) = \sum_{m=1}^2 G_m \Delta u_y(\xi) \quad (17)$$

The unknown function  $P_3(\xi)$  is approximated by the product of the fundamental function  $w_3(\xi)$  and the weight function  $F_3(\xi)$ .

$$P_3(\xi) = w_3(\xi) F_3(\xi) \quad (18)$$

By using the fundamental density function, the crack opening displacements may be expressed in the following equations<sup>[4]</sup>:

$$\Delta u_y = \sum_{m=1}^2 \frac{1}{G_m} w_3(\xi) F_3(\xi) = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m} \sqrt{a^2 - \xi^2} F_3(\xi) \quad (19)$$

The weight function  $F_3(\xi)$  is approximated by the following power series.

$$F_3(\xi) = \sum_{n=1}^M c_n \xi^{n-1} \quad (20)$$

A set of collocation points on the imaginary crack site is chosen as follows<sup>[6]</sup>:

$$x = a \cos\{n\pi / (M + 1)\}, \quad (n = 1, \dots, M), \quad (21)$$

where  $M$  is the number of the collocation points on the crack.

To satisfy the boundary condition, the points should be placed as close to the both ends of the cracks as shown in Eq.(21). Using the above method, the singular integral equation may be reduced to algebraic equations for the determination of coefficient  $c_n$ . The stress intensity factors are expressed from the unknown functions  $F_3(\xi)$  shown in Eq. (22).

$$K_3 = F_3(a) p_0 \sqrt{\pi a} \quad (22)$$

## 3. Results and discussion

### 3.1 Results for plane problems

Table 1 shows the convergence of stress intensity factors for the two-dimensional interface cracks with the varying number of polynomial  $M$  and the exponent  $n$ . It is seen that the present solutions give exact results for the exponent  $n$  when  $M \geq n + 1$ . Similarly, when the crack is subjected to shear stress, the present solutions also give exact

Table 1 Convergence of the results for 2D interface crack under polynomial distribution of stress

$$p_{zz} = p_0 (x/a)^n \quad (\varepsilon=0.02) \quad K_I + iK_{II} = \{F_I + iF_{II}\} p_0 \sqrt{\pi a} (1 + 2i\varepsilon)$$

	M	n=0	n=1	n=2	n=3	n=4	n=5	n=6
$F_1$	1	1.0000	0.0000	0.0000	0.0000	0.0000	0.00000	0.00000
	2	1.0000	0.5000	0.2500	0.1250	0.0625	0.03125	0.01563
	3	1.0000	0.5000	0.4997	0.2222	0.2221	0.09877	0.09872
	4	1.0000	0.5000	0.4997	0.3748	0.2772	0.21668	0.15567
	5	1.0000	0.5000	0.4997	0.3748	0.3747	0.24865	0.24855
	6	1.0000	0.5000	0.4997	0.3748	0.3747	0.31222	0.26921
	7	1.0000	0.5000	0.4997	0.3748	0.3747	0.31222	0.31212
	∞	1.0000	0.5000	0.4997	0.3748	0.3747	0.31222	0.31212
$F_2$	1	0.0000	0.0000	0.0000	0.00000	0.0000	0.00000	0.00000
	2	0.0000	0.0200	0.0000	0.00500	0.0000	0.00125	0.00000
	3	0.0000	0.0200	0.1333	0.00890	0.0059	0.00395	0.00263
	4	0.0000	0.0200	0.1333	0.01833	0.0083	0.01075	0.00474
	5	0.0000	0.0200	0.1333	0.01833	0.0147	0.01262	0.01037
	6	0.0000	0.0200	0.1333	0.01833	0.0147	0.01716	0.01159
	7	0.0000	0.0200	0.1333	0.01833	0.0147	0.01716	0.01471
	∞	0.0000	0.0200	0.1333	0.01833	0.0147	0.01716	0.01471

Table 2 Convergence of the results for 2D interface crack under polynomial distribution of stress

$$p_{xx} = p_0 (x/a)^n \quad (\varepsilon=0.02) \quad K_I + iK_{II} = \{F_I + iF_{II}\} p_0 \sqrt{\pi a} (1 + 2i\varepsilon)$$

M	n=0		n=1		n=2		n=3	
	$F_1'$	$F_2'$	$F_1'$	$F_2'$	$F_1'$	$F_2'$	$F_1'$	$F_2'$
1	0.0000	1.0000	0.0200	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	1.0000	0.0200	0.5000	0.0000	0.2500	0.0050	0.1250
3	0.0000	1.0000	0.0200	0.5000	0.1333	0.4997	0.0089	0.2222
4	0.0000	1.0000	0.0200	0.5000	0.1333	0.4997	0.0183	0.3748
5	0.0000	1.0000	0.0200	0.5000	0.1333	0.4997	0.0183	0.3748
∞	0.0000	1.0000	0.0200	0.5000	0.1333	0.4997	0.01833	0.3748

results when  $M \geq n+1$ . The stress intensity factors  $F_1', F_2'$  have been changed to be  $F_1' = F_2, F_2' = F_1$  (see Table.2).

Besides, when the interface is subjected to polynomial distribution of normal stress  $p_{zz} = p_0 (x/a)^n$ , the coefficients of polynomials have been examined (see Table 3).

When  $n$  is odd, that is,  $n=2N-1$  ( $N$  is a natural number), unknowns  $F_1(\xi)$  should be an even function while  $F_2(\xi)$  should be an odd function. Therefore, coefficients  $a_i \neq 0, b_i = 0$  when  $i$  is even number and  $i \leq n+1$ , and  $a_i = 0, b_i \neq 0$  when  $i$  is odd number and  $i \leq n+1$ . Those coefficients  $a_i = 0, b_i = 0$  when  $i > n+1$ . Also, when  $n$  is even number, that is  $n=2N$ , unknowns  $F_1(\xi)$  should be an odd function while  $F_2(\xi)$  should be an even function. Therefore, coefficients  $a_i \neq 0, b_i = 0$  when  $i$  is odd and  $i \leq n+1$ , and  $a_i = 0, b_i \neq 0$  when  $i$  is even number and  $i \leq n+1$ . Those coefficients  $a_j = 0, b_j = 0$  when  $i > n+1$ . From Table 3, we can propose the formula for the coefficients  $a_i, b_i$  as functions of  $\varepsilon$  as shown in Table 4.

Besides, in Table 5 we have proposed the stress intensity factor's formula for  $F_1, F_2$  when the interface crack is under polynomial distribution of tensile stress  $p_y = p_0(x/a)^n$  and shear stress  $p_x = p_0(x/a)^n$ . The results in Table 5 have more than 4 digit accuracy.

On the other hand, the crack opening displacements can be written in the following way.



Table 3 Coefficient of power series

		$\varepsilon = 0.02$	$\varepsilon = 0.06$	$\varepsilon = 0.10$
$a_n$	n=0	$a_1=1$	$a_1=1$	$a_1=1$
	n=1	$a_2=0.5$	$a_2=0.5$	$a_2=0.5$
	n=2	$a_1=0.1664, a_3=0.3333$	$a_1=0.1643, a_3=0.3333$	$a_1=0.1600, a_3=0.3333$
	n=3	$a_2=0.1248, a_4=0.25$	$a_2=0.1232, a_4=0.25$	$a_2=0.1200, a_4=0.25$
	n=4	$a_1=0.07481, a_3=0.09984, a_5=0.2$	$a_1=0.07332, a_3=0.09856, a_5=0.2000$	$a_1=0.07035, a_3=0.09600, a_5=0.2000$
	n=5	$a_2=0.06234, a_4=0.08320, a_6=0.1667$	$a_2=0.06110, a_4=0.08213, a_6=0.1667$	$a_2=0.05862, a_4=0.08000, a_6=0.1667$
	n=6	$a_1=0.04450, a_3=0.05344, a_5=0.07131, a_7=0.1429$	$a_1=0.04339, a_3=0.05237, a_5=0.07040, a_7=0.1429$	$a_1=0.04118, a_3=0.05025, a_5=0.06857, a_7=0.1429$
$b_n$	n=0	$b_0=0$	$b_0=0$	$b_0=0$
	n=1	$b_1=0.02$	$b_1=0.06$	$b_1=0.1$
	n=2	$b_2=0.01334$	$b_2=0.04$	$b_2=0.06667$
	n=3	$b_1=0.008332, b_3=0.01$	$b_1=0.02493, b_3=0.03$	$b_1=0.04133, b_3=0.05$
	n=4	$b_2=0.006666, b_4=0.008001$	$b_2=0.01994, b_4=0.024$	$b_2=0.03307, b_4=0.04$
	n=5	$b_0=0.004943, b_5=0.005555, b_5=0.006668$	$b_1=0.01476, b_3=0.01662, b_5=0.02$	$b_1=0.02439, b_3=0.02756, b_5=0.03333$
	n=6	$b_0=0.004237, b_0=0.004761, b_6=0.005715$	$b_2=0.01265, b_4=0.01425, b_6=0.01714$	$b_2=0.02091, b_4=0.02362, b_6=0.02857$

Table 4 General expression for coefficient of power series

	$a_n$	$b_n$
n=0	$a_1=1$	$b_0=0$
n=1	$a_2=0.5$	$b_1=\varepsilon$
n=2	$a_1=-0.6667\varepsilon^2+0.16667, a_3=0.333333$	$b_2=0.66667\varepsilon$
n=3	$a_2=-0.5\varepsilon^2+0.125, a_4=0.25$	$b_1=0.416667\varepsilon, b_3=0.5\varepsilon$
n=4	$a_1=-0.46611\varepsilon^2+0.075$ $a_3=-0.4\varepsilon^2+0.1, a_5=0.2$	$b_2=0.333333\varepsilon$ $b_4=0.4\varepsilon$
n=5	$a_2=-0.388\varepsilon^2+0.0625$ $a_4=-0.3334\varepsilon^2+0.0833333$ $a_6=0.166667$	$b_1=0.247135\varepsilon$ $b_3=0.27774\varepsilon$ $b_5=0.333333\varepsilon$
n=6	$a_1=-0.35\varepsilon^2+0.0446428$ $a_3=-0.332\varepsilon^2+0.0535714$ $a_5=-0.28782\varepsilon^2+0.071429$ $a_7=0.142857$	$b_2=0.21183\varepsilon$ $b_4=0.238065\varepsilon$ $b_6=0.28575\varepsilon$

(a) When the crack is under  $p_{zz} = p_0(x/a)^n$  and n is odd number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1+\kappa_m}{4G_m \cosh \pi\varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a-\xi}{a+\xi} \right)^{i\varepsilon} \times [a_2(\xi/a) + a_4(\xi/a)^3 + \dots + a_{n+1}(\xi/a)^n + i\{b_1 + b_3(\xi/a)^2 + \dots + b_n(\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx}) \quad (23.a)$$

(b) When the crack is under  $p_{zz} = p_0(x/a)^n$  and n is even number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_1 + a_3(\xi/a)^2 + \dots + a_{n+1}(\xi/a)^n + i\{b_2(\xi/a) + b_4(\xi/a)^3 + \dots + b_n(\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx}) \quad (23.b)$$

(c) When the crack is under  $p_{zx} = p_0(x/a)^n$  and n is odd number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_2(\xi/a) + a_4(\xi/a)^3 + \dots + a_{n+1}(\xi/a)^n + i\{b_1 + b_3(\xi/a)^2 + \dots + b_n(\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx}) \quad (24.a)$$

(d) When the crack is under  $p_{zy} = p_0(x/a)^n$  and n is even number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_2(\xi/a) + a_4(\xi/a)^3 + \dots + a_{n+1}(\xi/a)^n + i\{b_1 + b_3(\xi/a)^2 + \dots + b_n(\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx}) \quad (24.b)$$

Table 5 General expression of  $F_I, F_{II}$  under polynomial distribution of stress  $K_I + iK_{II} = \{F_I + iF_{II}\} p_0 \sqrt{\pi a} (1 + 2i\varepsilon)$

(a) tension

	$F_I$	$F_{II}$	$F_I (\varepsilon=0)$
n=0	1	0	1
n=1	0.5	$\varepsilon$	1/2=0.5
n=2	$0.5 - 0.6667 \varepsilon^2$	$(2/3) \varepsilon$	1/2=0.5
n=3	$0.375 - 0.5 \varepsilon^2$	$0.916 \varepsilon$	3/8=0.375
n=4	$0.375 - 0.8661 \varepsilon^2$	$0.733 \varepsilon$	3/8=0.375
n=5	$0.3125 - 0.7214 \varepsilon^2$	$0.857 \varepsilon$	5/16=0.3125
n=6	$0.3125 - 0.9698 \varepsilon^2$	$0.735 \varepsilon$	5/16=0.3125

(b) shear

	$F_I$	$F_{II}$	$F_{II} (\varepsilon=0)$
n=0	0	1	1
n=1	$\varepsilon$	0.5	1/2=0.5
n=2	$(2/3) \varepsilon$	$0.5 - 0.6667 \varepsilon^2$	1/2=0.5
n=3	$0.916 \varepsilon$	$0.375 - 0.5 \varepsilon^2$	3/8=0.375
n=4	$0.733 \varepsilon$	$0.375 - 0.8661 \varepsilon^2$	3/8=0.375
n=5	$0.857 \varepsilon$	$0.3125 - 0.7214 \varepsilon^2$	5/16=0.3125
n=6	$0.735 \varepsilon$	$0.3125 - 0.9698 \varepsilon^2$	5/16=0.3125

### 3.2 Results for transverse shear problem

When the crack is subjected to transverse shear, the crack opening displacement can be written in the following way.

(e) When n is odd number

$$\Delta u_y = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m} \sqrt{a^2 - \xi^2} \times \{c_2(\xi/a) + c_4(\xi/a)^3 + \dots + c_{n+1}(\xi/a)^n\} p_{zy} \quad (25.a)$$

(f) When n is even number

$$\Delta u_y = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m} \sqrt{a^2 - \xi^2} \times \{c_1 + c_3 (\xi/a)^2 + \dots + c_n (\xi/a)^{n-1}\} p_{zy} \quad (25.b)$$

Table 6 Coefficient  $c_n$  of power series

n=0	$c_1=1$
n=1	$c_2=0.5$
n=2	$c_1=0.1667, c_3=0.3333$
n=3	$c_2=0.125, c_4=0.25$
n=4	$c_1=0.075, c_3=0.1, c_5=0.2$
n=5	$c_2=0.0625, c_4=0.08333, c_6=0.1667$
n=6	$c_1=0.0446428, c_3=0.0535714,$ $c_5=0.0714285, c_7=0.142857$

Table 7 General expression of  $F_{III}$  under polynomial distribution of stress  $K_{III} = F_{III} p_0 \sqrt{\pi a}$

	$F_{III}$
n=0	1
n=1	1/2=0.5
n=2	1/2=0.5
n=3	3/8=0.375
n=4	3/8=0.375
n=5	5/16=0.3125
n=6	5/16=0.3125

The formula for the coefficients  $c_n$  is shown in Table 6, and the formula for  $F_3$  is shown in Table 7.

#### 4. Conclusions

In this study a two-dimensional interface crack under polynomial distribution of stress was investigated through hypersingular integral equations on the basis of the body force method. The general expression of the stress intensity factor under any combination of materials and the effect of the exponent n was discussed. The conclusions of this paper can be summarized as follows.

- (1) The present solution gives exact stress intensity factors for 2D interface crack under  $p_{zz} = p_0(x/a)^n$  and  $p_{zx} = p_0(x/a)^n$  when the number of polynomial  $M = n + 1$ .
- (2) The exact expression of crack opening displacements is expressed in the following equations when the crack is under polynomial distribution of stress.

(a) When the crack is under  $p_{zz} = p_0(x/a)^n$  and n is odd number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_2 (\xi/a) + a_4 (\xi/a)^3 + \dots + a_{n+1} (\xi/a)^n + i\{b_1 + b_3 (\xi/a)^2 + \dots + b_n (\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx})$$

(b) When the crack is under  $p_{zz} = p_0(x/a)^n$  and n is even number.



$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_1 + a_3 (\xi/a)^2 + \dots + a_{n+1} (\xi/a)^n + i\{b_2 (\xi/a) + b_4 (\xi/a)^3 + \dots + b_n (\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx})$$

(c) When the crack is under  $p_{zx} = p_0(x/a)^n$  and n is odd number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_2 (\xi/a) + a_4 (\xi/a)^3 + \dots + a_{n+1} (\xi/a)^n + i\{b_1 + b_3 (\xi/a)^2 + \dots + b_n (\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx})$$

(d) When the crack is under  $p_{zy} = p_0(x/a)^n$  and n is even number.

$$\Delta u_z + i\Delta u_x = \sum_{m=1}^2 \frac{1 + \kappa_m}{4G_m \cosh \pi \varepsilon} \sqrt{a^2 - \xi^2} \left( \frac{a - \xi}{a + \xi} \right)^{i\varepsilon} \times [a_2 (\xi/a) + a_4 (\xi/a)^3 + \dots + a_{n+1} (\xi/a)^n + i\{b_1 + b_3 (\xi/a)^2 + \dots + b_n (\xi/a)^{n-1}\}] \times (p_{zz} + ip_{zx})$$

The coefficients  $a_n, b_n$  are given in Table 3.

(3) For an interface crack under general pressure  $P_{zz}, P_{zx}, P_{zy}$ , the stress intensity factor is controlled by bimaterials parameter  $\varepsilon$  alone, whose expressions are given in Table 5 and 7.

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